

Modalized Domains on Possibility Frames*

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Abstract

Modalized domain models interpret higher-order modal logic over a Kripke frame by equipping each type with a family of world-indexed partial equivalence relations, which say what exists and what is identical at each world. We generalize these models from Kripke frames to possibility frames, indexing identity to partial possibilities ordered by refinement rather than to worlds. Full models of this kind validate Classicism together with strong comprehension and choice principles, yet their algebras of propositions need not be atomic. We use them to settle several questions left open in recent work on Classicism: the consistency of $\Box$ Rigid Comprehension and $\Box$ Intensional Choice with the failure of Atomicity (with and without the Barcan formula), the consistency of Mathematical Possibilism with these principles in models with arbitrarily many propositions, and the consistency of certain logical atomist principles with the failure of Atomicity.

Introduction

Classicism is the higher-order theory of propositional and relational granularity that results from adding to classical higher-order logic the identification of any two relations that are provably coextensive by the lights of that logic (Bacon, 2018a; Bacon and Dorr, 2024). Classicism is a natural minimal theory, but it leaves open a great many questions about the structure of propositions and properties. Are any distinct things necessarily distinct (ND)? Does the Barcan formula hold (BF)? Is every possible proposition entailed by a maximally specific possible proposition—a *world* proposition (Atomicity)? Is every property coextensive with a rigid property (RC)? Can one choose, from each necessarily instantiated property, a property entailing it that is necessarily uniquely instantiated (IC)? Much recent work has been devoted to mapping out which combinations of answers to these questions are consistent with Classicism.

The main tool for establishing such consistency results has been model theory. Bacon and Dorr (2024) define *action models*, in which propositions are sets of arrows in a category acting on the domains, and show that they are sound and complete for Classicism. Two earlier model

*The following note, and the verification of all of the theorems, was written up by Claude Opus 5.5 on October 19 2026. The first author had the general definition of a modalized domain on a possibility frame around 2020, and was using them to solve the conjecture about Mathematical Possibilism in December 2025. In the spring of 2026, Zach Goodsell reported that Claude Fable 5 had found an algebraic model of a related conjecture which was essentially equivalent. The remaining theorems are fairly simple applications of the framework suggested by the first author, which Claude wrote up.

theories, the *modalized domain models* of Bacon (2018a), based on a reflexive and transitive Kripke frame, and the *substitution structures* of Bacon (2019), can be thought of as special cases of this where the category is a preorder and a monoid respectively. This note extends the modalized domain models of the former in a natural way.

These models have a limitation that matters for the questions above. When they are *full*—when every intension is in the domain of propositions, and every function that respects identity is in the relevant functional domain—the singleton of any world is a proposition. So Atomicity and Actuality, and their necessitations, hold in every propositionally full model (Bacon and Dorr, 2024, §3.5). To refute Atomicity one has therefore had to pass to non-full models. But fullness is exactly what makes strong comprehension and choice principles such as Rigid Comprehension, Boolean Completeness and Intensional Choice easy to verify, and in non-full models they tend to fail. In Appendix D of Bacon and Dorr (2024), for example, every model in which Atomicity fails also fails Boolean Completeness, and so Rigid Comprehension; the authors note that they did not try very hard to find models in which these comprehension principles hold while Atomicity fails.

This paper removes the limitation by replacing the worlds of a Kripke frame with the partial *possibilities* of a possibility frame (Humberstone, 1981; Holliday, 2025). A possibility frame comes with a refinement ordering on possibilities as well as an accessibility relation, and propositions are the sets of possibilities that are *persistent* (closed under refinement) and *refinable* (they contain any possibility all of whose refinements can be refined into them). The resulting algebra of propositions is always a complete Boolean algebra, but it need not be atomic: an infinitely descending chain of refinements need not bottom out in a world. We index the partial equivalence relations of a modalized domain by possibilities rather than worlds, subject to the corresponding persistence and refinability conditions. Full models of this kind are still full, so comprehension and choice principles can be verified as before, but Atomicity can now fail in them.

We use these models to settle several open questions.

Theorem A (Section 4). $C + \Box RC + \Box IC + \Box BF + \neg \text{Atomicity} + \neg ND$ is consistent.

Theorem B (Section 5). $C + \Box RC + \Box IC + \neg \text{Atomicity} + \neg ND$ is consistent with the failure of BF at type t .

As far as we know, these are the first models of Classicism in which $\Box$ Rigid Comprehension and Intensional Choice hold while Atomicity fails, and they answer the question raised by Bacon and Dorr (2024, Appendix D) in a strong form. Theorem B shows that the Barcan formula plays no essential role. It needs a new idea: Rigid Comprehension and BF jointly imply $\Box BF$ (Bacon and Dorr, 2024), so in a model of $\Box RC$ the Barcan formula cannot fail merely contingently, and we will see that a failure at type t requires propositions that come into existence, which the components of Theorem A cannot supply.

Theorem C (Section 6). For every cardinal κ , there is a model of $C + \Box RC + \Box IC + \Box BF + \text{Mathematical Possibilism}$ with more than κ propositions.

Mathematical Possibilism says that every small complete Boolean algebra—one that is finite or has fewer elements than there are propositions—is isomorphic to the algebra of propositions

entailing some non-trivial proposition. [Bacon \(2026\)](#) shows that $\Box$ Rigid Comprehension, Intensional Choice and Mathematical Possibilism jointly entail the structural contingency of the continuum hypothesis, and conjectures that these principles are consistent in the background logic. [Theorem C](#) establishes the conjecture in the stronger setting of Classicism.

Theorem D ([Section 7](#)). *The Tractarian Truth-Table Postulate and Alethic Supervenience of [Bacon \(MS\)](#), and their necessitations, are consistent with $C + \Box RC + \Box BF + \neg \text{Atomicity}$.*

The model for [Theorem D](#) is a variant of the model for [Theorem A](#) in which every atom of the propositional algebra is modally accessible from every other. Its algebra of propositions is one of the complete Boolean algebras that [Fritz and Bacon \(MS\)](#) show to be completely independently generated without being atomic.

All four models share one architecture. At the root sits a point that sees everything. Below it lie one or more *components*: in the simplest case, the refinement order of a complete atomless Boolean algebra, each of whose points sees only its own refinements. Within such a component the box is idle, every property is rigid, and the component behaves like a Boolean-valued model of extensional higher-order logic. At the root, by contrast, identity is literal and the actual extension of a property can be chosen independently of its behaviour in the components. Rigid Comprehension holds because a property can be rigidified by replacing its behaviour in the components with the “check-closure” of its actual extension, in close analogy with the check-names of forcing; Intensional Choice holds because choice within a component is Boolean mixing. The atomless components refute Atomicity.

Plan. [Section 1](#) introduces rooted possibility frames and their propositions. [Section 2](#) introduces modalized domains indexed by the points of a possibility frame, and defines models and the interpretation of the higher-order language in them. [Section 3](#) proves that Classicism is sound for these models, and that full models validate several further principles, including $\Box$ Rigid Comprehension and $\Box$ Intensional Choice under conditions we identify. [Sections 4 to 7](#) prove [Theorems A to D](#) respectively.

1 Possibility frames

1.1 Frames

We follow the presentation of possibility semantics in [Holliday \(2025\)](#), restricted to the full frames we need. Points of a frame are *possibilities*, which may be partial; $x' \sqsubseteq x$ says that x' *refines* x , that is, x' settles everything x settles and possibly more. More specific possibilities lie lower in the order. We write $\downarrow x := \{x' : x' \sqsubseteq x\}$ for the set of refinements of x , and $R[x] := \{y : xRy\}$ for the set of possibilities accessible from x . Two possibilities are *compatible* if they have a common refinement.

Definition 1.1. A *possibility frame* is a triple $\mathcal{F} = \langle S, \sqsubseteq, R \rangle$, where S is a non-empty set, $\sqsubseteq$ is a partial order on S , and R is a binary relation on S satisfying the following three conditions:

up- R if $x' \sqsubseteq x$ and $x'Ry$, then xRy ;

R-down if xRy and $y' \sqsubseteq y$, then xRy' ;

R-refinability if xRy , then there is an $x' \sqsubseteq x$ such that for every $x'' \sqsubseteq x'$ there is a $y' \sqsubseteq y$ with $x''Ry'$.

$\mathcal{F}$ is an S4 *possibility frame* if, in addition, R is reflexive and transitive.

The first two conditions are taken from Holliday (2025, Example 2.7). Up- R says that a coarser possibility sees at least what its refinements see; R -down says that the set of possibilities accessible from a given possibility is closed under refinement. R -refinability says that whenever x sees y , some refinement of x is such that every further refinement of it sees some refinement of y . Together, up- R and R -refinability imply the conditions that Holliday (2025, Prop. 2.30) shows to be necessary and sufficient for the box of R to preserve propositions; we give a direct proof of what we need in Proposition 1.6.

Since Classicism proves every theorem of S4 for its defined necessity operator (Bacon, 2018a), all the frames we use will be S4 possibility frames.

Definition 1.2. A *root* of a possibility frame $\mathcal{F}$ is a point $r \in S$ that sees every point: rRx for every $x \in S$. A *rooted possibility frame* is a pair $\langle \mathcal{F}, r \rangle$ of an S4 possibility frame and a root of it.

The root represents actuality, and will be the point at which truth is evaluated. We do not require the root to be $\sqsubseteq$ -minimal. If it is, truth at the root is bivalent (Lemma 1.8). If it is not, the root is a partial possibility, and some sentences may be neither true nor false there; this does no harm in consistency proofs (Corollary 3.7). Nor does the requirement that the root see everything cost any generality. Every point x of an S4 possibility frame is a root of the subframe generated by x : the set $R[x]$, with the restrictions of $\sqsubseteq$ and R . This set is closed under refinement and accessibility, by R -down and transitivity, so it is again an S4 possibility frame, and only this subframe matters for evaluation at x . So any point of any S4 possibility frame may serve as a point of evaluation.

1.2 Propositions

Definition 1.3. Let $\mathcal{F} = \langle S, \sqsubseteq, R \rangle$ be a possibility frame. A set $X \subseteq S$ is

- *persistent* if $x \in X$ and $x' \sqsubseteq x$ imply $x' \in X$;
- *refinable* if, whenever every $x' \sqsubseteq x$ has a refinement $x'' \sqsubseteq x'$ in X , $x \in X$.

A *proposition* of $\mathcal{F}$ is a set that is both persistent and refinable. We write $\text{RO}(\mathcal{F})$ for the set of propositions of $\mathcal{F}$, and say that X *holds at* x when $x \in X$.

Persistence says that what holds at a possibility holds at all of its refinements. Refinability says that if a possibility cannot be refined to one where X fails to be settled true—every refinement of it can be further refined to one at which X holds—then X already holds at it. In topological terms, the propositions are exactly the regular open sets of the topology on S whose open sets are the downward closed sets (Holliday, 2025, Fact 2.17), which explains the notation. The following is standard.

Proposition 1.4. $\text{RO}(\mathcal{F})$, ordered by inclusion, is a complete Boolean algebra, with top S , bottom $\emptyset$, and

$$\begin{aligned}\bigwedge_k X_k &= \bigcap_k X_k, & \neg X &= \{x : \text{no } x' \sqsubseteq x \text{ is in } X\}, & \bigvee_k X_k &= \neg \bigcap_k \neg X_k, \\ X \rightarrow Y &= \neg X \vee Y = \{x : \text{every } x' \sqsubseteq x \text{ in } X \text{ is in } Y\}.\end{aligned}$$

Proof. In the topology on S whose open sets are the downward closed sets, the interior of Y is $\{x : \downarrow x \subseteq Y\}$ and the closure of X is $\{x : \downarrow x \cap X \neq \emptyset\}$. So X is persistent iff it is open, and a persistent X is refinable iff it contains the interior of its closure. The propositions are thus the regular open sets, which form a complete Boolean algebra with the displayed operations in any topological space (Givant and Halmos, 2009, ch. 10). For the formula for $X \rightarrow Y$: $x \in \neg(X \cap \neg Y)$ iff no refinement of x lies in $X \cap \neg Y$. If every refinement of x in X is in Y this holds. Conversely, if it holds and $x' \sqsubseteq x$ is in X , then every refinement of x' is in X and so not in $\neg Y$, so has a refinement in Y ; hence $x' \in Y$ by refinability. $\square$

Definition 1.5. For $X \subseteq S$, let $\Box_R X := \{x \in S : R[x] \subseteq X\}$.

Proposition 1.6. Let $\mathcal{F}$ be a possibility frame. Then $\Box_R X \in \text{RO}(\mathcal{F})$ whenever $X \in \text{RO}(\mathcal{F})$, and $\Box_R \bigcap_k X_k = \bigcap_k \Box_R X_k$ for any family of propositions. If $\mathcal{F}$ is an S4 possibility frame, then $\Box_R X \subseteq X$ and $\Box_R X \subseteq \Box_R \Box_R X$ for every X ; so $\Box_R$ is an interior operator on $\text{RO}(\mathcal{F})$ commuting with arbitrary meets.

Proof. Preservation of meets is immediate from the definition, and the S4 properties follow from reflexivity and transitivity of R as usual. Let X be a proposition.

Persistence. Let $x \in \Box_R X$, $x' \sqsubseteq x$ and $x' R y$. By up- R , $x R y$, so $y \in X$.

Refinability. Suppose every $x' \sqsubseteq x$ has a refinement in $\Box_R X$, and let $x R y$; we show $y \in X$, using the refinability of X . Let $y' \sqsubseteq y$. By R -down, $x R y'$. By R -refinability, there is $x' \sqsubseteq x$ such that every $x'' \sqsubseteq x'$ sees some refinement of y' . By hypothesis some $x'' \sqsubseteq x'$ is in $\Box_R X$; choose $z \sqsubseteq y'$ with $x'' R z$. Then $z \in X$. So every refinement of y has a refinement in X , and $y \in X$. $\square$

We collect some facts about S4 possibility frames that we will use repeatedly.

Lemma 1.7. Let $\mathcal{F}$ be an S4 possibility frame.

- (a) If $x' \sqsubseteq x$ then $x R x'$. Hence every set of the form $R[x]$ is downward closed, and contains $\downarrow x$.
- (b) For propositions X and Y : $x \in \Box_R(X \leftrightarrow Y)$ iff $X \cap R[x] = Y \cap R[x]$.
- (c) For propositions X and Y : $x \in \Box_R(X \rightarrow Y)$ iff $X \cap R[x] \subseteq Y$.

Proof. (a) $x R x$ by reflexivity, so $x R x'$ by R -down; and $R[x]$ is downward closed by R -down. (c) If $x \in \Box_R(X \rightarrow Y)$ and $y \in X \cap R[x]$, then $y \in (X \rightarrow Y) \cap X \subseteq Y$. Conversely, suppose $X \cap R[x] \subseteq Y$ and $y \in R[x]$. Every refinement of y lies in $R[x]$ by (a), so every refinement of y in X is in Y , and $y \in X \rightarrow Y$ by Proposition 1.4. (b) follows from (c), since $X \leftrightarrow Y = (X \rightarrow Y) \cap (Y \rightarrow X)$ and $\Box_R$ preserves intersections. $\square$

Lemma 1.8 (The root). Let r be a root of $\mathcal{F}$. For every proposition X , $r \in \Box_R X$ iff $X = S$. If, moreover, r is $\sqsubseteq$ -minimal, then truth at r is bivalent: for propositions X and X_k , $r \in \neg X$ iff $r \notin X$; $r \in \bigcap_k X_k$ iff $r \in X_k$ for every k ; and $r \in \bigvee_k X_k$ iff $r \in X_k$ for some k .

Proof. The first claim holds because $R[r] = S$. If r is $\sqsubseteq$ -minimal then $\downarrow r = \{r\}$, so $r \in \neg X$ iff $r \notin X$. The clause for $\cap$ is immediate, and the clause for $\vee = \neg \cap \neg$ follows from the other two. $\square$

1.3 Examples

Example 1.9 (Kripke frames). If $\sqsubseteq$ is the identity relation, the three conditions of [Definition 1.1](#) hold trivially, every subset of S is a proposition, and $\Box_R$ is the usual Kripke box. So an S4 Kripke frame is an S4 possibility frame in which every point is $\sqsubseteq$ -minimal. In such a frame every singleton $\{x\}$ is a proposition, and an atom of $\text{RO}(\mathcal{F}) = \mathcal{P}(S)$.

Example 1.10 (Forcing frames). Let $\langle P, \leq \rangle$ be a partial order, and let $\sqsubseteq$ be $\leq$ and xRy iff $y \leq x$: each point sees exactly its own refinements. The three conditions hold (for R -refinability, given xRy take $x' = y$, and let each $x'' \sqsubseteq y$ see itself), and R is reflexive and transitive. Since propositions are persistent, $\Box_R X = X$ for every proposition: the box is idle. $\text{RO}(\mathcal{F})$ is the regular open algebra of P ; if P is the set of non-zero elements of a complete Boolean algebra B , then $\text{RO}(\mathcal{F}) \cong B$, via $X \mapsto \bigvee X$ (cf. [Holliday, 2025](#)). In particular $\text{RO}(\mathcal{F})$ is atomless whenever B is. If P has a greatest element, that element sees every point, and so is a root; it is $\sqsubseteq$ -minimal only if it is the only point of P .

The frames of our main constructions are obtained by placing a root above a family of frames of this kind.

Definition 1.11 (Rooted sums). Let $\mathcal{F}_i = \langle S_i, \sqsubseteq_i, R_i \rangle$, for $i \in I$, be S4 possibility frames with pairwise disjoint sets of points, and let r be a new point. The *rooted sum* $\bigoplus_{i \in I} \mathcal{F}_i$ is the frame $\langle S, \sqsubseteq, R \rangle$ where

- $S = \{r\} \cup \bigcup_i S_i$;
- $\sqsubseteq$ is the union of the $\sqsubseteq_i$ together with $r \sqsubseteq r$;
- R is the union of the R_i together with $\{r\} \times S$.

The $\mathcal{F}_i$ are its *components*.

So the root sees every point, while a point of a component sees only what it sees in its own component; in particular no point of a component sees the root.

Proposition 1.12. *The rooted sum $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ is an S4 possibility frame, and r is a $\sqsubseteq$ -minimal root of it. A set $X \subseteq S$ is a proposition of $\mathcal{F}$ iff $X \cap S_i$ is a proposition of $\mathcal{F}_i$ for every i , so that*

$$\text{RO}(\mathcal{F}) \cong 2 \times \prod_{i \in I} \text{RO}(\mathcal{F}_i), \quad X \mapsto ([r \in X], (X \cap S_i)_{i \in I}),$$

and, for every proposition X ,

$$\Box_R X = \bigcup_i \Box_{R_i} (X \cap S_i) \cup \{r : X = S\}.$$

Proof. Up- R , R -down and R -refinability hold within each component, and hold trivially for pairs involving r , since r has no proper refinements and sees every point. Reflexivity and transitivity are clear, as is the fact that r is a root. Persistence and refinability of a set relate a

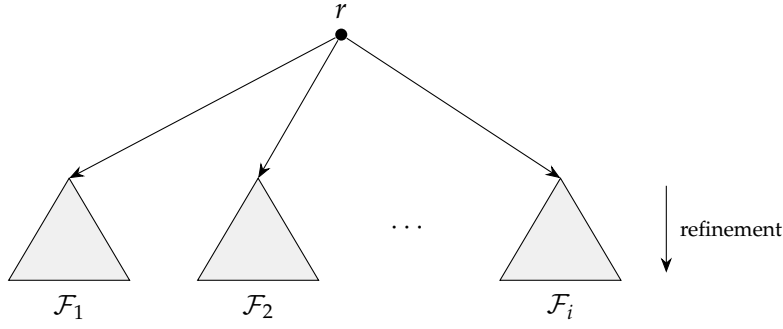


Figure 1: A rooted sum. The root r sees every point; a point of a component sees only points of its own component.

point only to its refinements, which lie in the same component as the point (or are r itself), so they can be checked componentwise; this gives the description of $\text{RO}(\mathcal{F})$. The formula for $\Box_R$ is immediate from the definition of R . $\square$

When each component is a forcing frame (Example 1.10), the box is idle on the components and Proposition 1.12 says that $\Box_R X$ agrees with X in each component, and holds at r iff X is the top proposition. If some component is atomless, so is the corresponding factor of $\text{RO}(\mathcal{F})$, and then $\text{RO}(\mathcal{F})$ is not atomic. These frames underlie the models of Sections 4 and 6.

2 Modalized domain models

2.1 The language

We work with the language of Bacon (2018a), restricted to the relational types; the only difference is that the type system of Bacon (2018a) also contains types ending in e , such as $e \rightarrow e$, which we will not need. Our types are thus given by the relational type system of Bacon and Dorr (2024): e and t are types, and $\sigma \rightarrow \tau$ is a type whenever σ and τ are and τ is not e . Every type other than e therefore has the form $\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t$ with $n \geq 0$; we call these the *relational* types. Terms are built from typed variables and constants by application and λ -abstraction. The logical constants are $\neg$ of type $t \rightarrow t$; $\wedge$ and $\vee$ of type $t \rightarrow t \rightarrow t$; and, for each type σ , quantifiers $\forall_\sigma$ and $\exists_\sigma$ of type $(\sigma \rightarrow t) \rightarrow t$ and an identity predicate $=_\sigma$ of type $\sigma \rightarrow \sigma \rightarrow t$. We may also add non-logical constants from a typed signature Σ . We use the abbreviations of Bacon and Dorr (2024, Fig. 1); in particular

$$\top := \forall p p \vee \neg \forall p p, \quad \Box := \lambda p. p =_t (p \vee \neg p), \quad \leq_\tau := \lambda XY. Y =_\tau X \vee_\tau Y.$$

H is classical higher-order logic, with β - and η -conversion, and *Classicism*, C , is the smallest H -theory containing every instance of the schema of Logical Equivalence: $(\lambda \vec{v}. A) = (\lambda \vec{v}. B)$ whenever $A \leftrightarrow B$ is a theorem of H (Bacon and Dorr, 2024, §1).

2.2 Modalized domains

For the rest of this section, fix a rooted possibility frame $\langle \mathcal{F}, r \rangle$ with $\mathcal{F} = \langle S, \sqsubseteq, R \rangle$. A partial equivalence relation (PER) on a set is a symmetric and transitive relation on it; it is an equivalence relation on the set of elements it relates to themselves.

Definition 2.1. A *modalized domain* over $\mathcal{F}$ is a pair $\mathbf{D} = \langle |\mathbf{D}|, (\sim_x)_{x \in S} \rangle$, where $|\mathbf{D}|$ is a set and each $\sim_x$ is a PER on $|\mathbf{D}|$, satisfying:

Persistence if $a \sim_x b$ and xRy , then $a \sim_y b$;

Refinability if every $x' \sqsubseteq x$ has a refinement $x'' \sqsubseteq x'$ with $a \sim_{x''} b$, then $a \sim_x b$.

We write $\mathbf{D}_x := \{a \in |\mathbf{D}| : a \sim_x a\}$, and say that a *exists at* x , or is *available at* x , if $a \in \mathbf{D}_x$.

Informally, $a \sim_x b$ says that a and b exist at x and are identical there. Persistence says that existence and identity are preserved along accessibility; since $x' \sqsubseteq x$ implies xRx' (Lemma 1.7), they are in particular preserved under refinement. Refinability is the analogue, for identity, of the refinability of propositions. When $\sqsubseteq$ is the identity relation, refinability is trivial, and a modalized domain is exactly a modalized domain in the sense of Bacon (2018a). As in that setting, domains may expand along accessibility, and distinct things may become identical, but nothing ceases to exist and identical things never become distinct.

Definition 2.2 (Function spaces). If $\mathbf{D}$ and $\mathbf{E}$ are modalized domains over $\mathcal{F}$, the modalized domain $\mathbf{D} \Rightarrow \mathbf{E}$ has as its underlying set the set of all functions from $|\mathbf{D}|$ to $|\mathbf{E}|$, and

$$f \sim_x g \quad \text{iff} \quad \text{for every } y \in R[x] \text{ and all } a, b \in |\mathbf{D}|, \text{ if } a \sim_y b \text{ then } fa \sim_y gb.$$

So f and g are identical at x iff they necessarily map identical arguments to identical values. In particular f exists at x iff it necessarily respects identity.

Lemma 2.3. $\mathbf{D} \Rightarrow \mathbf{E}$ is a modalized domain.

Proof. *PER.* Symmetry is inherited from the PERs of $\mathbf{D}$ and $\mathbf{E}$. For transitivity, suppose $f \sim_x g$ and $g \sim_x h$, and let $y \in R[x]$ and $a \sim_y b$. Then $b \sim_y b$, so $fa \sim_y gb$ and $gb \sim_y hb$, and hence $fa \sim_y hb$.

Persistence. If $f \sim_x g$ and xRy , then every $z \in R[y]$ is in $R[x]$ by transitivity of R , so $f \sim_y g$.

Refinability. Suppose every refinement of x has a refinement at which $f \sim g$. Let $y \in R[x]$ and $a \sim_y b$; we must show $fa \sim_y gb$, and by refinability in $\mathbf{E}$ it suffices to show that every $y' \sqsubseteq y$ has a refinement w with $fa \sim_w gb$. By R -down, xRy' . By R -refinability, there is $x' \sqsubseteq x$ such that every $x'' \sqsubseteq x'$ sees some refinement of y' . By hypothesis some $x'' \sqsubseteq x'$ has $f \sim_{x''} g$; choose $w \sqsubseteq y'$ with $x''Rw$. Since $w \sqsubseteq y$, we have yRw (Lemma 1.7), so persistence in $\mathbf{D}$ gives $a \sim_w b$; and since $x''Rw$ and $f \sim_{x''} g$, $fa \sim_w gb$. $\square$

The proof uses refinability only at the codomain and persistence only at the domain; in particular, refinability need never be imposed on functional types, as it is inherited from the base types.

Definition 2.4 (The propositional domain). $\mathbf{P}_{\mathcal{F}}$ is the modalized domain whose underlying set is $\text{RO}(\mathcal{F})$, with

$$X \sim_x Y \quad \text{iff} \quad X \cap R[x] = Y \cap R[x].$$

By [Lemma 1.7](#), $X \sim_x Y$ iff $x \in \Box_R(X \leftrightarrow Y)$: two propositions are identical at x iff they are necessarily equivalent there. Every proposition exists at every point of $\mathbf{P}_{\mathcal{F}}$.

Lemma 2.5. $\mathbf{P}_{\mathcal{F}}$ is a modalized domain.

Proof. Each $\sim_x$ is an equivalence relation, and persistence follows from transitivity of R , since $R[y] \subseteq R[x]$ when xRy . For refinability, suppose every refinement of x has a refinement at which $X \sim Y$. Let $u \in R[x] \cap X$; we show $u \in Y$ (the converse inclusion is symmetric). Let $u' \sqsubseteq u$. Then xRu' by R -down, so by R -refinability there is $x' \sqsubseteq x$ such that every $x'' \sqsubseteq x'$ sees some refinement of u' . Choose $x'' \sqsubseteq x'$ with $X \sim_{x''} Y$, and $w \sqsubseteq u'$ with $x''Rw$. Then $w \in X$ by persistence of X , so $w \in Y$. So every refinement of u has a refinement in Y , and $u \in Y$ by refinability of Y . $\square$

The models we consider need not contain every function, or let every proposition exist everywhere. As in [Bacon \(2018a\)](#), this is captured by a notion of subdomain.

Definition 2.6. A modalized domain $\mathbf{D}$ is a *subdomain* of $\mathbf{E}$, written $\mathbf{D} \leq \mathbf{E}$, if $|\mathbf{D}| \subseteq |\mathbf{E}|$, $\mathbf{D}_x \subseteq \mathbf{E}_x$ for every x , and for all $a, b \in \mathbf{D}_x$, $a \sim_x^{\mathbf{D}} b$ iff $a \sim_x^{\mathbf{E}} b$.

For example, given a family $(A_x)_{x \in S}$ of sets of propositions, the relations $X \sim_x Y$ iff $X, Y \in A_x$ and $X \cap R[x] = Y \cap R[x]$ define a subdomain of $\mathbf{P}_{\mathcal{F}}$, provided that they satisfy persistence and refinability; this is so iff $A_x \subseteq A_y$ whenever xRy , and $X \in A_x$ whenever every refinement of x has a refinement x'' with $X \in A_{x''}$. Here A_x is the set of propositions available at x . Subdomains of this kind will be used in [Section 5](#) to make new propositions come into existence.

2.3 Structures, models and interpretation

Definition 2.7. A *modalized domain structure* over $\langle \mathcal{F}, r \rangle$ assigns to each type σ a modalized domain $\mathbf{D}^\sigma$ over $\mathcal{F}$, such that

- (i) $\mathbf{D}_x^e \neq \emptyset$ for every $x \in S$;
- (ii) $\mathbf{D}^t \leq \mathbf{P}_{\mathcal{F}}$;
- (iii) $\mathbf{D}^{\sigma \rightarrow \tau} \leq \mathbf{D}^\sigma \Rightarrow \mathbf{D}^\tau$ for all types $\sigma \rightarrow \tau$.

The structure is *functionally full* if $\mathbf{D}^{\sigma \rightarrow \tau} = \mathbf{D}^\sigma \Rightarrow \mathbf{D}^\tau$ for every functional type, *propositionally full* if $\mathbf{D}^t = \mathbf{P}_{\mathcal{F}}$, and *full* if it is both.

A full structure is determined by the frame together with the modalized domain $\mathbf{D}^e$ of individuals. Condition (i) reflects the fact that $\exists x (x = x)$ is a theorem of H at every type.

Next we define the operations that will interpret the logical constants. For propositions X, Y , functions $f: |\mathbf{D}^\sigma| \rightarrow \text{RO}(\mathcal{F})$, and $a, b \in |\mathbf{D}^\sigma|$, let

$$\begin{aligned} \text{neg}(X) &:= \neg X, & \text{and}(X)(Y) &:= X \cap Y, & \text{or}(X)(Y) &:= X \vee Y, \\ \text{all}^\sigma(f) &:= \{x \in S : x' \in f(a) \text{ for every } x' \sqsubseteq x \text{ and every } a \in \mathbf{D}_{x'}^\sigma\}, \\ \text{some}^\sigma(f) &:= \neg \text{all}^\sigma(\text{neg} \circ f), \\ \text{id}^\sigma(a)(b) &:= \{x \in S : a \sim_x^\sigma b\}. \end{aligned}$$

$\text{all}^\sigma(f)$ holds at x iff, however x is refined, f holds at that refinement of everything that exists there. The quantification over refinements is needed because refinements may have larger domains than the possibilities they refine; when $\mathbf{D}_{x'}^\sigma = \mathbf{D}_x^\sigma$ for all $x' \sqsubseteq x$, it holds at x iff $x \in f(a)$ for every $a \in \mathbf{D}_x^\sigma$, by persistence of the propositions $f(a)$. When $\sqsubseteq$ is the identity, all^σ is the universal quantifier of [Bacon \(2018a\)](#).

Lemma 2.8. *The values of neg, and, or, all^σ , some^σ and id^σ are propositions.*

Proof. For the Boolean operations this is [Proposition 1.4](#), and some^σ is defined from all^σ and $\neg$. The set $\text{id}^\sigma(a)(b)$ is persistent because $x' \sqsubseteq x$ implies xRx' and $\sim$ is persistent, and refinable by refinability of $\sim$. $\text{all}^\sigma(f)$ is persistent because $\sqsubseteq$ is transitive. For refinability, suppose every refinement of x has a refinement in $\text{all}^\sigma(f)$, and let $x' \sqsubseteq x$ and $a \in \mathbf{D}_{x'}^\sigma$. Every $x'' \sqsubseteq x'$ has a refinement $z \in \text{all}^\sigma(f)$; then $a \in \mathbf{D}_z^\sigma$ by persistence, so $z \in f(a)$. So every refinement of x' has a refinement in $f(a)$, and $x' \in f(a)$. $\square$

An *assignment* is a function g mapping each variable of type σ to an element of $|\mathbf{D}^\sigma|$. It is an *x -assignment* if $g(v) \in \mathbf{D}_x^\sigma$ for every variable v of type σ ; that is, if it assigns to each variable something that exists at x .

Definition 2.9. A *modalized domain model* $\mathcal{M}$ is a modalized domain structure over a rooted possibility frame, together with an interpretation I assigning to each constant $c \in \Sigma$ of type σ an element $I(c) \in |\mathbf{D}^\sigma|$ that exists at every point, such that the following clauses define, for each term A of type σ and each assignment g , an element $\llbracket A \rrbracket_g \in |\mathbf{D}^\sigma|$:

$$\begin{aligned} \llbracket v \rrbracket_g &= g(v), & \llbracket c \rrbracket_g &= I(c) \ (c \in \Sigma), \\ \llbracket \neg \rrbracket_g &= \text{neg}, & \llbracket \wedge \rrbracket_g &= \text{and}, & \llbracket \vee \rrbracket_g &= \text{or}, \\ \llbracket \forall_\sigma \rrbracket_g &= \text{all}^\sigma, & \llbracket \exists_\sigma \rrbracket_g &= \text{some}^\sigma, & \llbracket =_\sigma \rrbracket_g &= \text{id}^\sigma, \\ \llbracket AB \rrbracket_g &= \llbracket A \rrbracket_g(\llbracket B \rrbracket_g), & \llbracket \lambda v. A \rrbracket_g &= (d \mapsto \llbracket A \rrbracket_{g[v \mapsto d]}). \end{aligned}$$

That is, we require that neg, and, or, all^σ , some^σ and id^σ belong to the underlying sets of the relevant domains, and that the function $d \mapsto \llbracket A \rrbracket_{g[v \mapsto d]}$ belongs to $|\mathbf{D}^{\sigma \rightarrow \tau}|$ for every term $\lambda v. A$ of type $\sigma \rightarrow \tau$ and every assignment g . We further require that each of the logical operations exists at every point of S :

$$\text{neg, and, or, all}^\sigma, \text{some}^\sigma, \text{id}^\sigma \in \mathbf{D}_x \text{ for every } x \in S.$$

A formula is a term of type t , so $\llbracket A \rrbracket_g$ is a proposition when A is a formula.

Definition 2.10. Let $\mathcal{M}$ be a modalized domain model over $\langle \mathcal{F}, r \rangle$. A formula A holds at x under g if $x \in \llbracket A \rrbracket_g$. A sentence A is true in $\mathcal{M}$ if $r \in \llbracket A \rrbracket_g$ (for any, equivalently every, assignment g). A formula A is valid in $\mathcal{M}$ if $x \in \llbracket A \rrbracket_g$ for every $x \in S$ and every x -assignment g .

Validity quantifies only over x -assignments, as in any semantics with variable domains: an open formula such as $x = x$ is valid even though it may fail at points where the value of x does not exist. As we will see in Section 3, the theorems of Classicism are valid in every modalized domain model. Since the root may be a partial possibility, a sentence may be neither true nor false in a model. But the sentences true in a model always belong to a consistent H-theory extending Classicism (Corollary 3.7), which is all that consistency proofs require; and when the root is $\sqsubseteq$ -minimal, the true sentences form a complete theory (Lemma 2.13).

Proposition 2.11. Let $\langle \mathcal{F}, r \rangle$ be a rooted possibility frame, $\mathbf{D}^e$ a modalized domain over $\mathcal{F}$ with $\mathbf{D}_x^e \neq \emptyset$ for every x , and I an interpretation of Σ in the full structure determined by $\mathbf{D}^e$, assigning to each constant something that exists everywhere. Then the full structure, together with I , is a modalized domain model.

Proof. Since every function between the underlying sets belongs to the underlying set of the relevant function space, and every proposition belongs to $|\mathbf{D}^t| = \text{RO}(\mathcal{F})$, the clauses of Definition 2.9 define $\llbracket A \rrbracket_g$ for every term and assignment. It remains to show that the logical operations exist at every point. We first unfold the definition of existence at relational types. Repeatedly applying Definition 2.2, and using that R is reflexive and transitive and that $\sim$ is persistent, an induction on n shows that a function F in the underlying set of $\mathbf{D}^{\sigma_1 \rightarrow \dots \rightarrow \sigma_n \rightarrow t}$ exists at x iff

$$\text{for all } y \in R[x] \text{ and all } a_i \sim_y b_i \ (1 \leq i \leq n) : \quad F(a_1) \cdots (a_n) \cap R[y] = F(b_1) \cdots (b_n) \cap R[y]. \quad (*)$$

We verify $(*)$ for each operation; let $y \in R[x]$.

- **neg:** if $X \cap R[y] = Y \cap R[y]$ and $z \in R[y]$, then all refinements of z lie in $R[y]$ (Lemma 1.7), so $z \in \neg X$ iff $z \in \neg Y$. The case of **and** is immediate, and **or** = **neg** $\circ$ **and** $\circ$ (**neg** $\times$ **neg**).
- **all $^\sigma$:** suppose $f \sim_y f'$ and $z \in R[y]$. For $z' \sqsubseteq z$ we have $z' \in R[y]$, so $f \sim_{z'} f'$ by persistence, and for $a \in \mathbf{D}_{z'}^\sigma$, $f(a) \sim_{z'} f'(a)$; since $z' \in R[z']$, $z' \in f(a)$ iff $z' \in f'(a)$. Hence $z \in \text{all}^\sigma(f)$ iff $z \in \text{all}^\sigma(f')$. The case of **some $^\sigma$** follows.
- **id $^\sigma$:** if $a \sim_y a'$ and $b \sim_y b'$, then for $z \in R[y]$ we have $a \sim_z a'$ and $b \sim_z b'$ by persistence, so $a \sim_z b$ iff $a' \sim_z b'$. $\square$

In models that are functionally but not propositionally full, the first part of the argument still applies, and the second does too, provided that the values of the logical operations on available arguments are available; this is what must be checked in Section 5.

Finally we record how the defined notions behave.

Lemma 2.12. Let $\mathcal{M}$ be a modalized domain model, $x \in S$, and $X \in \mathbf{D}_x^t$. Then $x \in \llbracket \Box p \rrbracket_g$ iff $x \in \Box_R X$, where $g(p) = X$. In words: on propositions that exist at x , the defined necessity operator of Classicism holds at x exactly when the box of the accessibility relation does.

Proof. If f exists at x and a exists at x , then $f(a)$ exists at x , by [Definition 2.2](#) with $y = x$. Since neg , or X exist at x , so does $\llbracket p \vee \neg p \rrbracket_g = X \vee \neg X = S$. So $x \in \llbracket \Box p \rrbracket_g$ iff $X \sim_x^t S$, and since $X, S \in \mathbf{D}_x^t$ and $\mathbf{D}^t \leq \mathbf{P}_{\mathcal{F}}$, this holds iff $X \cap R[x] = R[x]$, that is, iff $x \in \Box_R X$. $\square$

Lemma 2.13. *Let $\mathcal{M}$ be a modalized domain model over $\langle \mathcal{F}, r \rangle$, and let g be an r -assignment. Then (c) and (d) below hold. If r is $\sqsubseteq$ -minimal, (a) and (b) hold as well, and for every sentence A , exactly one of A and $\neg A$ is true in $\mathcal{M}$.*

- (a) $r \in \llbracket \neg A \rrbracket_g$ iff $r \notin \llbracket A \rrbracket_g$; $r \in \llbracket A \wedge B \rrbracket_g$ iff $r \in \llbracket A \rrbracket_g$ and $r \in \llbracket B \rrbracket_g$; and similarly for $\vee$;
- (b) $r \in \llbracket \forall v A \rrbracket_g$ iff $r \in \llbracket A \rrbracket_{g[v \mapsto a]}$ for every $a \in \mathbf{D}_r^\sigma$; and similarly for $\exists$;
- (c) $r \in \llbracket A = B \rrbracket_g$ iff $\llbracket A \rrbracket_g \sim_r \llbracket B \rrbracket_g$;
- (d) $r \in \llbracket \Box A \rrbracket_g$ iff $\llbracket A \rrbracket_g = S$, provided $\llbracket A \rrbracket_g \in \mathbf{D}_r^t$.

Proof. (c) is the definition of id^σ , and (d) follows from [Lemma 2.12](#) and [Lemma 1.8](#). If r is $\sqsubseteq$ -minimal, (a) is [Lemma 1.8](#); for (b), $\downarrow r = \{r\}$, so $r \in \text{all}^\sigma(f)$ iff $r \in f(a)$ for every $a \in \mathbf{D}_r^\sigma$, and the clause for $\exists$ follows using (a). $\square$

So, at a $\sqsubseteq$ -minimal root, a modalized domain model looks like a Tarskian model whose domain of type σ is $\mathbf{D}_r^\sigma$ modulo $\sim_r$; the modal operator quantifies over the whole frame. If moreover $\sim_r$ is the identity relation on each $\mathbf{D}_r^\sigma$, as in all our examples, identity at the root is literal. All the models of [Sections 4 to 7](#) have $\sqsubseteq$ -minimal roots.

2.4 Examples

Example 2.14 (Kripke frames). When $\sqsubseteq$ is the identity relation, the persistence and refinability conditions on propositions and on modalized domains are trivial (beyond persistence along R), $\text{RO}(\mathcal{F})$ is the full powerset of S , and the operation all^σ quantifies, at each world, over what exists there. So modalized domain models over Kripke frames are exactly the modalized domain models of [Bacon \(2018a\)](#). In a propositionally full model over a Kripke frame, the singleton of the root is a proposition that exists at the root, so Atomicity and Actuality hold there; this is the obstacle discussed in the introduction.

Example 2.15 (Boolean-valued models). Let B be a complete atomless Boolean algebra, and let $\mathcal{F}$ be the forcing frame on $B \setminus \{0\}$ ([Example 1.10](#)). Its greatest element 1_B is a root, though not a $\sqsubseteq$ -minimal one. Take $|\mathbf{D}^e|$ to be a set D , with $a \sim_x b$ iff $a = b$ for every x , and consider the full structure. Since the box is idle, $X \sim_x^t Y$ iff $x \in X \leftrightarrow Y$, and identity of propositions at x coincides with their material equivalence there. Identifying $\text{RO}(\mathcal{F})$ with B , the structure is, in effect, the full B -valued model of extensional higher-order logic over the domain D , in which the Boolean value of $a = b$ is $\{x : a \sim_x b\}$. From the inside, every proposition is identical to the true or the false one at every point: $X \vee \neg X = S$, and $x \in X$ implies $X \sim_y S$ for all refinements y of x . In particular, Extensionality holds throughout such a structure. With 1_B as the root, a sentence is true in the model iff its Boolean value is 1_B , so the usual Boolean-valued models of extensional higher-order logic are a special case of our models; in them, of course, a sentence may be neither true nor false. Such frames will also serve as components of rooted frames whose roots are $\sqsubseteq$ -minimal.

The models of [Sections 4, 6 and 7](#) are full models over rooted sums whose components are forcing frames of this kind; the model of [Section 5](#) adds a component in which the box is not idle and new propositions come into existence.

3 Soundness, and principles valid in full models

Throughout this section, $\mathcal{M}$ is a modalized domain model over a rooted possibility frame $\langle \mathcal{F}, r \rangle$, with $\mathcal{F} = \langle S, \sqsubseteq, R \rangle$. For tuples $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b}$ of entities of types $\vec{\sigma}$, we write $\vec{a} \sim_x \vec{b}$ if $a_i \sim_x b_i$ for each i , $\vec{a} \in \mathbf{D}_x$ if each a_i exists at x , and $\text{id}(\vec{a})(\vec{b}) := \bigcap_i \text{id}^{\sigma_i}(a_i)(b_i)$. Quantifiers over tuples abbreviate strings of quantifiers in the usual way.

3.1 Three lemmas about denotation

Lemma 3.1 (Coincidence, substitution, replacement).

- (a) If g and h agree on the free variables of A , then $\llbracket A \rrbracket_g = \llbracket A \rrbracket_h$.
- (b) If B is free for v in A , then $\llbracket A[B/v] \rrbracket_g = \llbracket A \rrbracket_{g[v \mapsto \llbracket B \rrbracket_g]}$.
- (c) If $\llbracket A \rrbracket_h = \llbracket A' \rrbracket_h$ for every assignment h , and $\Phi[A']$ results from $\Phi[A]$ by replacing an occurrence of A with A' , then $\llbracket \Phi[A] \rrbracket_g = \llbracket \Phi[A'] \rrbracket_g$ for every assignment g .

Proof. Standard inductions on terms. They use only the fact that the denotation of an application, or of an abstraction $\lambda v.A$, is computed from the denotations of its immediate subterms under the given assignment or its v -variants. $\square$

The next lemma is the analogue for our models of the fundamental lemma of logical relations. It says that terms respect identity at every point.

Lemma 3.2 (Fundamental lemma). *Let $x \in S$, and let g and h be assignments with $g(v) \sim_x h(v)$ for every variable v . Then $\llbracket A \rrbracket_g \sim_x \llbracket A \rrbracket_h$ for every term A . In particular, $\llbracket A \rrbracket_g$ exists at x whenever g is an x -assignment, and the denotation of a closed term exists at every point.*

Proof. By induction on A , simultaneously for all points and all pairs of assignments. For variables this is the hypothesis. Constants, logical and non-logical, exist at every point by [Definition 2.9](#). For an application AB , suppose $f \sim_x f'$ and $a \sim_x a'$; since xRx , [Definition 2.2](#) gives $f(a) \sim_x f'(a')$. For an abstraction $\lambda v.A$ with v of type σ , let $y \in R[x]$ and $d \sim_y d'$. By persistence, $g(u) \sim_y h(u)$ for every variable u , so $g[v \mapsto d]$ and $h[v \mapsto d']$ are related pointwise at y , and the induction hypothesis at y gives $\llbracket A \rrbracket_{g[v \mapsto d]} \sim_y \llbracket A \rrbracket_{h[v \mapsto d']}$. By [Definition 2.2](#), $\llbracket \lambda v.A \rrbracket_g \sim_x \llbracket \lambda v.A \rrbracket_h$. $\square$

Lemma 3.3 (Minimal points). *Let w be a $\sqsubseteq$ -minimal point. For propositions X and X_k , and $f: |\mathbf{D}^\sigma| \rightarrow \text{RO}(\mathcal{F}): w \in \neg X$ iff $w \notin X$; $w \in \bigvee_k X_k$ iff $w \in X_k$ for some k ; $w \in X \rightarrow Y$ iff $w \notin X$ or $w \in Y$; and $w \in \text{all}^\sigma(f)$ iff $w \in f(a)$ for every $a \in \mathbf{D}_w^\sigma$.*

Proof. As for [Lemmas 1.8 and 2.13](#), using $\downarrow w = \{w\}$. $\square$

3.2 Soundness

Theorem 3.4. *Every theorem of Classicism is valid in every modalized domain model.*

Let $\mathcal{V}$ be the set of formulas valid in $\mathcal{M}$. Since Classicism is the smallest H-theory containing every instance of Logical Equivalence, it suffices to show that $\mathcal{V}$ contains the axioms of H, is closed under its rules, and contains every instance of Logical Equivalence. We use the axiomatization of H in [Bacon \(2018a\)](#): the axioms PC, UI, EG, Ref, LL, β and η , and the rules MP, Gen and Inst. Recall from [Proposition 1.4](#) that $x \in X \rightarrow Y$ iff every refinement of x in X is in Y .

Proof of Theorem 3.4: the axioms. Fix $x \in S$ and an x -assignment g . By [Lemma 3.2](#), the denotation under g of every term exists at x , and hence, by persistence, at every refinement of x .

PC. The connectives denote the Boolean operations of $\text{RO}(\mathcal{F})$, so every tautology denotes S under every assignment.

UI: $\forall_\sigma F \rightarrow FA$. Let $f = \llbracket F \rrbracket_g$ and $a = \llbracket A \rrbracket_g$. If $x' \sqsubseteq x$ is in $\text{all}^\sigma(f)$, then since a exists at x' , $x' \in f(a)$.

EG: $FA \rightarrow \exists_\sigma F$. Suppose $x' \sqsubseteq x$ is in $f(a)$ but not in $\text{some}^\sigma(f) = \neg \text{all}^\sigma(\text{neg} \circ f)$. Then some $x'' \sqsubseteq x'$ is in $\text{all}^\sigma(\text{neg} \circ f)$, so $x'' \in \neg f(a)$, since a exists at x'' . But $x'' \in f(a)$ by persistence, a contradiction.

Ref: $A = A$. $\llbracket A \rrbracket_g$ exists at x , so $x \in \text{id}(\llbracket A \rrbracket_g)(\llbracket A \rrbracket_g)$.

LL: $A = B \rightarrow (FA \rightarrow FB)$. Let $a = \llbracket A \rrbracket_g$, $b = \llbracket B \rrbracket_g$ and $f = \llbracket F \rrbracket_g$, and let $x' \sqsubseteq x$ with $a \sim_{x'} b$. Since f exists at x' , $f(a) \sim_{x'} f(b)$, so $f(a)$ and $f(b)$ agree on $R[x'] \supseteq \downarrow x'$. Hence every refinement of x' in $f(a)$ is in $f(b)$, and $x' \in f(a) \rightarrow f(b)$.

β and η . By [Lemma 3.1\(b\)](#), $\llbracket (\lambda v.A)B \rrbracket_h = \llbracket A \rrbracket_{h[v \mapsto \llbracket B \rrbracket_h]} = \llbracket A[B/v] \rrbracket_h$ for every assignment h ; and $\llbracket \lambda v.Fv \rrbracket_h = \llbracket F \rrbracket_h$ when v is not free in F , since both are the function $d \mapsto \llbracket F \rrbracket_h(d)$. By [Lemma 3.1\(c\)](#), $\Phi[(\lambda v.A)B]$ and $\Phi[A[B/v]]$ (respectively $\Phi[\lambda v.Fv]$ and $\Phi[F]$) have the same denotation under every assignment, so their biconditional denotes S . $\square$

Proof of Theorem 3.4: the rules. MP. If $x \in \llbracket P \rightarrow Q \rrbracket_g$ and $x \in \llbracket P \rrbracket_g$, then $x \in \llbracket Q \rrbracket_g$.

Gen. Suppose $P \rightarrow Q \in \mathcal{V}$ and v is not free in P . Let g be an x -assignment, and let $x' \sqsubseteq x$ be in $\llbracket P \rrbracket_g$; we show $x' \in \text{all}^\sigma(f)$, where $f = \llbracket \lambda v.Q \rrbracket_g$. Let $x'' \sqsubseteq x'$ and $a \in \mathbf{D}_{x''}^\sigma$. Then $g[v \mapsto a]$ is an x'' -assignment, by persistence, and $x'' \in \llbracket P \rrbracket_g = \llbracket P \rrbracket_{g[v \mapsto a]}$. By the validity of $P \rightarrow Q$, $x'' \in \llbracket Q \rrbracket_{g[v \mapsto a]} = f(a)$.

Inst. Suppose $P \rightarrow Q \in \mathcal{V}$ and v is not free in Q . Let g be an x -assignment, $f = \llbracket \lambda v.P \rrbracket_g$, and suppose for contradiction that $x' \sqsubseteq x$ is in $\text{some}^\sigma(f)$ but not in $\llbracket Q \rrbracket_g$. By refinability of $\llbracket Q \rrbracket_g$, some $x'' \sqsubseteq x'$ has no refinement in $\llbracket Q \rrbracket_g$. We show $x'' \in \text{all}^\sigma(\text{neg} \circ f)$, contradicting $x'' \in \text{some}^\sigma(f)$. Let $z \sqsubseteq x''$ and $a \in \mathbf{D}_z^\sigma$. If some $z' \sqsubseteq z$ were in $f(a) = \llbracket P \rrbracket_{g[v \mapsto a]}$, then, as $g[v \mapsto a]$ is a z' -assignment, the validity of $P \rightarrow Q$ would put z' in $\llbracket Q \rrbracket_{g[v \mapsto a]} = \llbracket Q \rrbracket_g$; but $z' \sqsubseteq x''$. So $z \in \neg f(a)$. $\square$

It remains to verify Logical Equivalence, which we derive from a more general fact.

Lemma 3.5. *If $A \leftrightarrow B \in \mathcal{V}$, where A and B are formulas, then $(\lambda \vec{v}.A) = (\lambda \vec{v}.B) \in \mathcal{V}$ for every sequence of variables $\vec{v}$.*

Proof. Say that terms C and D of the same type are *equivalent* if $\llbracket C \rrbracket_h \sim_y \llbracket D \rrbracket_h$ for every $y \in S$ and every y -assignment h . Then $C = D$ is valid iff C and D are equivalent. We show that A and B are equivalent, and that if C and D are equivalent then so are $\lambda v.C$ and $\lambda v.D$.

For the first claim, let h be a y -assignment. Both $\llbracket A \rrbracket_h$ and $\llbracket B \rrbracket_h$ exist at y , by Lemma 3.2, so it suffices that they agree on $R[y]$. If $z \in R[y]$, then h is a z -assignment, so $z \in \llbracket A \leftrightarrow B \rrbracket_h$; and since $z \in \llbracket A \rrbracket_h$ and $z \in \llbracket A \rrbracket_h \rightarrow \llbracket B \rrbracket_h$ imply $z \in \llbracket B \rrbracket_h$, and conversely, $z \in \llbracket A \rrbracket_h$ iff $z \in \llbracket B \rrbracket_h$.

For the second claim, let h be a y -assignment, $z \in R[y]$, and $d \sim_z d'$. Then $h[v \mapsto d]$ is a z -assignment, so $\llbracket C \rrbracket_{h[v \mapsto d]} \sim_z \llbracket D \rrbracket_{h[v \mapsto d]}$ by hypothesis; and $\llbracket D \rrbracket_{h[v \mapsto d]} \sim_z \llbracket D \rrbracket_{h[v \mapsto d']}$ by Lemma 3.2. By transitivity, $\llbracket \lambda v.C \rrbracket_h \sim_y \llbracket \lambda v.D \rrbracket_h$. $\square$

Since every theorem of H is in $\mathcal{V}$ by the first two parts of the proof, Lemma 3.5 shows that $\mathcal{V}$ contains every instance of Logical Equivalence. This completes the proof of Theorem 3.4. Note also that $\mathcal{V}$ is closed under Necessitation: if A is valid, g is an x -assignment and $y \in R[x]$, then g is a y -assignment, so $y \in \llbracket A \rrbracket_g$; hence $x \in \Box_R \llbracket A \rrbracket_g$, and $x \in \llbracket \Box A \rrbracket_g$ by Lemma 2.12.

When the root has proper refinements, the formulas that hold at the root under every r -assignment need not be closed under Gen: the quantifiers at the root look at its refinements, where new entities may exist, and a variant of an r -assignment that assigns such an entity is not an r -assignment. The following notion avoids this.

Definition 3.6. A formula A is *r -valid* in $\mathcal{M}$ if $x \in \llbracket A \rrbracket_g$ for every $x \sqsubseteq r$ and every x -assignment g .

Corollary 3.7. Let $\mathcal{M}$ be a model over $\langle \mathcal{F}, r \rangle$.

- (a) The r -valid formulas form an H -theory containing every theorem of Classicism, and $\perp$ is not r -valid.
- (b) A sentence is r -valid iff it is true in $\mathcal{M}$.
- (c) Hence, if every member of a set Γ of sentences is true in $\mathcal{M}$, the smallest H -theory containing Classicism and Γ is consistent, and every sentence in it is true in $\mathcal{M}$.
- (d) If r is $\sqsubseteq$ -minimal, then for every sentence A , exactly one of A and $\neg A$ is true in $\mathcal{M}$.
- (e) If A is valid in $\mathcal{M}$, then so is $\Box A$.

Proof. (a) The theorems of Classicism are valid, by Theorem 3.4, and so r -valid. The proofs of MP, Gen and Inst in the proof of Theorem 3.4 appeal to the validity of the premises only at refinements of the point under consideration, under assignments of entities existing there, so they show equally that the r -valid formulas are closed under these rules. And $\perp$ denotes $\emptyset$. (b) The denotation of a sentence does not depend on the assignment (Lemma 3.1(a)), and r -assignments exist, since $\mathbf{D}_r^e$ is non-empty and every other domain contains the denotation of a closed term, which exists everywhere. So if $r \in \llbracket A \rrbracket$ then $x \in \llbracket A \rrbracket$ for every $x \sqsubseteq r$, by persistence; and conversely $r \sqsubseteq r$. (c) follows from (a) and (b), and (d) is Lemma 2.13. (e) is the remark above. $\square$

3.3 World–cloud frames

All of our models are built over frames of the following special kind.

Definition 3.8. A rooted possibility frame $\langle \mathcal{F}, r \rangle$ is a *world–cloud frame* if S is partitioned into sets W and T such that

- (i) every $w \in W$ is $\sqsubseteq$ -isolated: w is $\sqsubseteq$ -comparable with no point other than itself;
- (ii) every $t \in T$ is *transparent*: $R[t] = \downarrow t$.

The points of W are the *worlds*, and those of T form the *cloud*.

A Kripke frame with a root is a world–cloud frame with an empty cloud, and a rooted sum of forcing frames (Example 1.10 and Definition 1.11) is a world–cloud frame whose only world is its root. The root may also be a cloud point; since it is transparent and sees every point, the frame is then a forcing frame with a greatest element, as in Example 2.15. Worlds may see one another and the cloud in any way compatible with the S4 and frame conditions, but each cloud point sees exactly its own refinements, so no cloud point sees a world.

Lemma 3.9. Let $\langle \mathcal{F}, r \rangle$ be a world–cloud frame.

- (a) T is downward closed, $R[t] \subseteq T$ for $t \in T$, and $R[w] \cap T$ is downward closed for $w \in W$.
- (b) A set $X \subseteq S$ is a proposition iff $X \cap T$ is a proposition of the subframe T ; so $\text{RO}(\mathcal{F}) \cong \mathcal{P}(W) \times \text{RO}(T)$.
- (c) (Transparency.) If $t \in T$, g is a t -assignment and A is a formula, then $t \in \llbracket \Box A \rrbracket_g$ iff $t \in \llbracket A \rrbracket_g$.

Proof. (a) A refinement of a cloud point is not a world, by isolation, so lies in the cloud. The other claims follow, using Lemma 1.7. (b) Persistence and refinability relate a point only to its refinements, and a world has none but itself. (c) $\llbracket A \rrbracket_g$ exists at t by Lemma 3.2, so by Lemma 2.12 $t \in \llbracket \Box A \rrbracket_g$ iff $\downarrow t \subseteq \llbracket A \rrbracket_g$, iff $t \in \llbracket A \rrbracket_g$ by persistence. $\square$

So, inside the cloud, the box is idle: the cloud behaves like a Boolean-valued model of extensional higher-order logic (Example 2.15). The worlds, by contrast, behave like the worlds of a Kripke model: by Lemma 3.3, truth at a world is bivalent and its quantifiers range over what exists there.

For a point x , let $\langle x \rangle := \neg\neg\downarrow x$. This is the smallest proposition containing x : its members are the points every refinement of which is compatible with x , and every proposition containing x contains $\langle x \rangle$. If x and y are incompatible, no refinement of y is in $\langle x \rangle$. If w is a world, $\langle w \rangle = \{w\}$.

3.4 Rigid Comprehension

Following Bacon (2026) and Bacon and Dorr (2024), for X of relational type $\vec{\sigma} \rightarrow t$ let

$$\begin{aligned} \text{Rig } X &:= \Box \forall Y^{\vec{\sigma} \rightarrow t} (\Box \forall \vec{z} (X\vec{z} \rightarrow Y\vec{z}) \leftrightarrow \forall \vec{z} (X\vec{z} \rightarrow \Box Y\vec{z})), \\ \text{RC}_{\vec{\sigma}} &:= \forall R^{\vec{\sigma} \rightarrow t} \exists X^{\vec{\sigma} \rightarrow t} (\text{Rig } X \wedge \forall \vec{z} (R\vec{z} \leftrightarrow X\vec{z})). \end{aligned}$$

The right-to-left direction of the biconditional in Rig says that X is *inextensible*: whatever necessarily holds of each actual X necessarily holds of every possible X . The left-to-right direction says that X is *persistent*: actual X s are necessarily X . Rigid Comprehension (RC) is the schema of the $\text{RC}_{\vec{\sigma}}$, one for each relational type.

Lemma 3.10 (Everything is rigid in the cloud). *Let $\mathcal{M}$ be a model over a world–cloud frame, t a cloud point, and $X \in \mathbf{D}_t^{\vec{\sigma} \rightarrow t}$. Then $t \in \llbracket \text{Rig } X \rrbracket$.*

Proof. Write $\psi(Y)$ for $\Box A \leftrightarrow B$, where A is $\forall \vec{z}(X\vec{z} \rightarrow Y\vec{z})$ and B is $\forall \vec{z}(X\vec{z} \rightarrow \Box Y\vec{z})$. Since $R[t] = \downarrow t$ and every point of $\downarrow t$ is a cloud point, it suffices to show that every $u \in \downarrow t$ lies in $\llbracket \psi(Y) \rrbracket$ for every Y existing at u , under the obvious assignment; it then follows that $u \in \llbracket \forall Y \psi(Y) \rrbracket$ for each such u , and that $t \in \llbracket \Box \forall Y \psi(Y) \rrbracket$. By [Proposition 1.4](#), $u \in \llbracket \psi(Y) \rrbracket$ iff $\llbracket \Box A \rrbracket$, $\llbracket A \rrbracket$ and $\llbracket B \rrbracket$ agree on $\downarrow u$, and it suffices that they agree there. For $\llbracket \Box A \rrbracket$ and $\llbracket A \rrbracket$, this is [Lemma 3.9\(c\)](#) applied at each point of $\downarrow u$. For $\llbracket A \rrbracket$ and $\llbracket B \rrbracket$: by [Lemma 3.9\(c\)](#), for each $u' \sqsubseteq u$ and each $\vec{z}$ existing at u' , the propositions $Y\vec{z}$ and $\Box Y\vec{z}$ agree on $\downarrow u'$, hence so do $X\vec{z} \rightarrow Y\vec{z}$ and $X\vec{z} \rightarrow \Box Y\vec{z}$. Since membership of a point u' in $\text{all}(f)$ depends only on the values of f on $\downarrow u'$ at arguments existing there, $\llbracket A \rrbracket$ and $\llbracket B \rrbracket$ agree on $\downarrow u$. $\square$

Definition 3.11 (Check-closure). For a point w and a set E of tuples of entities of types $\vec{\sigma}$, the *check-closure* of E is the function

$$\check{E}(\vec{z}) := \bigvee_{\vec{y} \in E} \text{id}(\vec{y})(\vec{z}) \quad (\text{join in } \text{RO}(\mathcal{F})).$$

So $\check{E}$ holds of $\vec{z}$ at a point to the extent that $\vec{z}$ is identical there to some member of E . When w is a world and $E \subseteq \mathbf{D}_w$, $\check{E}$ holds of $\vec{z}$ at w iff $\vec{z} \sim_w \vec{y}$ for some $\vec{y} \in E$, by [Lemma 3.3](#). The name is chosen by analogy with the check-names of forcing.

Theorem 3.12. *Let $\mathcal{M}$ be a full model over a world–cloud frame. Then $\text{RC}_{\vec{\sigma}}$ is valid in $\mathcal{M}$ for every relational type $\vec{\sigma} \rightarrow t$. Hence so is $\Box \text{RC}_{\vec{\sigma}}$.*

Proof. Let $x \in S$ and $R \in \mathbf{D}_x^{\vec{\sigma} \rightarrow t}$; we find a rigid X existing at x and coextensive with R there.

Case 1: x is a cloud point. Take $X = R$, which is rigid at x by [Lemma 3.10](#); $\forall \vec{z}(R\vec{z} \leftrightarrow X\vec{z})$ holds everywhere.

Case 2: x is a world w . Let $E = \{\vec{y} \in \mathbf{D}_w : w \in R\vec{y}\}$ be the extension of R at w , and let $X = \check{E}$.

X exists at w . By the unfolding (*) in the proof of [Proposition 2.11](#), we must show that if $v \in R[w]$ and $\vec{z} \sim_v \vec{z}'$, then $X\vec{z}$ and $X\vec{z}'$ agree on $R[v]$. Each $\text{id}(\vec{y})(\vec{z})$ agrees with $\text{id}(\vec{y})(\vec{z}')$ on $R[v]$, by persistence and transitivity of $\sim$; and joins preserve agreement on the downward closed set $R[v]$, since $\bigvee = \neg \bigcap \neg$ and membership of a point in $\neg Y$ depends only on Y below that point.

X is coextensive with R at w . For $\vec{z}$ existing at w , $w \in X\vec{z}$ iff $\vec{z} \sim_w \vec{y}$ for some $\vec{y} \in E$, iff $w \in R\vec{z}$, since R respects identity at w .

X is rigid at w . We must show that every $u \in R[w]$ is in $\llbracket \forall Y \psi(Y) \rrbracket$, with ψ as in the proof of [Lemma 3.10](#). If u is a cloud point, this is [Lemma 3.10](#). So let u be a world in $R[w]$, and let Y exist at u . Since u is $\sqsubseteq$ -minimal, $u \in \llbracket \forall Y \psi(Y) \rrbracket$ iff $u \in \llbracket \psi(Y) \rrbracket$ for every such Y , and by [Lemmas 2.12](#) and [3.3](#), $u \in \llbracket \Box A \rrbracket$ iff (L) and $u \in \llbracket B \rrbracket$ iff (R), where

(L) $X\vec{z} \cap R[u] \subseteq Y\vec{z}$ for every $\vec{z}$;

(R) $R[u] \subseteq Y\vec{z}$ for every $\vec{z} \in \mathbf{D}_u$ with $u \in X\vec{z}$.

(For (L) we use that $R[u]$ is downward closed, refinability, and the fact that $\vec{z}$ exists at every point of $X\vec{z}$, by refinability of $\sim$.) So it suffices to show that (L) and (R) are equivalent. (L) $\Rightarrow$ (R):

if $u \in X\vec{z}$ then $\vec{z} \sim_u \vec{y}$ for some $\vec{y} \in E$, and by persistence $R[u] \subseteq \text{id}(\vec{y})(\vec{z}) \subseteq X\vec{z}$; so $R[u] \subseteq Y\vec{z}$ by (L). (R) $\Rightarrow$ (L): each $\vec{y} \in E$ exists at u , by persistence, and $u \in X\vec{y}$, so $R[u] \subseteq Y\vec{y}$ by (R). Now let $v \in R[u] \cap \text{id}(\vec{y})(\vec{z})$. Since Y exists at u , $Y\vec{y} \sim_v Y\vec{z}$, and $v \in Y\vec{y}$, so $v \in Y\vec{z}$. Hence $R[u] \cap \text{id}(\vec{y})(\vec{z}) \subseteq Y\vec{z}$ for every $\vec{y} \in E$. Since $Y\vec{z}$ is refinable and $R[u]$ is downward closed, it follows that $R[u] \cap X\vec{z} \subseteq Y\vec{z}$.

So RC holds at every point under every assignment, and is valid; $\Box$ RC is then valid by [Corollary 3.7](#). $\square$

For Kripke frames with roots, [Theorem 3.12](#) recovers the familiar fact that full modalized domain models validate $\Box$ RC.

The proof also shows that, at a world, the rigid relations are exactly the check-closures of their extensions.

Proposition 3.13. *Let $\mathcal{M}$ be a full model over a world–cloud frame, w a world, and X a relation that exists and is rigid at w , with extension E at w . Then $X \sim_w \check{E}$.*

Proof. Both exist at w , and both respect identity at every point of $R[w]$, so it suffices to show that $X\vec{z} \cap R[w] = \check{E}\vec{z} \cap R[w]$ for every $\vec{z}$. The characterization of $\psi(Y)$ at a world by the conditions (L) and (R) in the proof of [Theorem 3.12](#) did not depend on the choice of X , so we may apply it at $u = w$. With $Y = \check{E}$, (R) holds, since $\vec{z} \in E$ implies $R[w] \subseteq \text{id}(\vec{z})(\vec{z}) \subseteq \check{E}\vec{z}$; so (L) gives $X\vec{z} \cap R[w] \subseteq \check{E}\vec{z}$. With $Y = X$, (L) holds trivially, so (R) gives $R[w] \subseteq X\vec{y}$ for every $\vec{y} \in E$. If $v \in R[w] \cap \text{id}(\vec{y})(\vec{z})$, then $X\vec{y} \sim_v X\vec{z}$, so $v \in X\vec{z}$; by refinability, $\check{E}\vec{z} \cap R[w] \subseteq X\vec{z}$. $\square$

3.5 Intensional Choice

Let $\exists! \vec{x} G\vec{x}$ abbreviate $\exists \vec{x}(G\vec{x} \wedge \forall \vec{y}(G\vec{y} \rightarrow \vec{y} = \vec{x}))$, and let $\text{IC}_{\vec{\sigma}}$ be

$$\forall F^{\vec{\sigma} \rightarrow t} \left(\Box \exists \vec{x} F\vec{x} \rightarrow \exists G^{\vec{\sigma} \rightarrow t} (\Box \forall \vec{x}(G\vec{x} \rightarrow F\vec{x}) \wedge \Box \exists! \vec{x} G\vec{x}) \right).$$

Intensional Choice (IC) is the schema of the $\text{IC}_{\vec{\sigma}}$ ([Bacon, 2026](#)).

Theorem 3.14. *Let $\mathcal{M}$ be a full model over a world–cloud frame. Then $\text{IC}_{\vec{\sigma}}$ is valid in $\mathcal{M}$ for every relational type $\vec{\sigma} \rightarrow t$. Hence so is $\Box \text{IC}_{\vec{\sigma}}$.*

Proof. Let $x \in S$ and $F \in \mathbf{D}_x$, and suppose $R[x] \subseteq \llbracket \exists \vec{x} F\vec{x} \rrbracket$. We construct G . Write $W_x = R[x] \cap W$ and $T_x = R[x] \cap T$; the latter is downward closed ([Lemma 3.9](#)).

Choice at the worlds. Each $v \in W_x$ is in $\llbracket \exists \vec{x} F\vec{x} \rrbracket$, so by [Lemma 3.3](#) we may choose $\vec{x}_v \in \mathbf{D}_v$ with $v \in F\vec{x}_v$.

Choice in the cloud. Call a point $u \in T_x$ *decided* if $u \in F\vec{x}$ for some $\vec{x} \in \mathbf{D}_u$. By Zorn’s lemma, choose a maximal family $(u_k)_{k \in K}$ of pairwise incompatible decided points of T_x , together with witnesses $\vec{x}_k \in \mathbf{D}_{u_k}$ with $u_k \in F\vec{x}_k$. Every point of T_x is compatible with some u_k . For if $u \in T_x$, then $u \in \text{some}(F)$, so $u \notin \text{all}(\text{neg} \circ F)$, and some $u' \sqsubseteq u$ and $\vec{x} \in \mathbf{D}_{u'}$ have $u' \notin \neg F\vec{x}$; hence some $u'' \sqsubseteq u'$ is in $F\vec{x}$, and is decided, and so is compatible with some u_k by maximality.

The witness. Let

$$G\vec{z} := \{v \in W_x : \vec{z} \sim_v \vec{x}_v\} \cup \bigvee_{k \in K} (\langle u_k \rangle \cap \text{id}(\vec{x}_k)(\vec{z})).$$

The first part is a set of worlds and the second a proposition contained in T (worlds are compatible with no cloud point), so $G\vec{z}$ is a proposition by Lemma 3.9(b). G exists at x : if $v \in R[x]$ and $\vec{z} \sim_v \vec{z}'$, then $\vec{z} \sim_{v'} \vec{z}'$ for all $v' \in R[v]$, so the conditions defining the two parts give the same answer for $\vec{z}$ and $\vec{z}'$ at every world in $R[v]$, and the cloud parts agree on $R[v] \cap T$ as in the proof of Theorem 3.12.

G entails F at x . We show $G\vec{z} \cap R[x] \subseteq F\vec{z}$ for all $\vec{z}$, which gives $x \in \llbracket \Box \forall \vec{x} (G\vec{x} \rightarrow F\vec{x}) \rrbracket$ as in the proof of Theorem 3.12. At a world $v \in W_x$: if $\vec{z} \sim_v \vec{x}_v$ then $F\vec{z} \sim_v F\vec{x}_v \ni v$, so $v \in F\vec{z}$. In the cloud, let $q \in T_x \cap \langle u_k \rangle \cap \text{id}(\vec{x}_k)(\vec{z})$; every refinement of q has a common refinement q' with u_k , at which $q' \in F\vec{x}_k$ and $\vec{x}_k \sim_{q'} \vec{z}$, so $q' \in F\vec{z}$; by refinability $q \in F\vec{z}$. The join over k is handled by refinability of $F\vec{z}$.

G is necessarily uniquely instantiated at x . At a world $v \in W_x$, $\vec{x}_v$ witnesses $\exists! \vec{x} G\vec{x}$, by Lemma 3.3: $v \in G\vec{x}_v$, and if $\vec{y} \in \mathbf{D}_v$ and $v \in G\vec{y}$ then $\vec{y} \sim_v \vec{x}_v$. In the cloud, let $u \in T_x$; by refinability it suffices to show that every $u' \sqsubseteq u$ has a refinement at which $\exists! \vec{x} G\vec{x}$ holds. Choose k and a common refinement q of u' and u_k . Every refinement q' of q refines u_k , so is in $\langle u_k \rangle$ and in no $\langle u_l \rangle$ with $l \neq k$. Hence $G\vec{y} \cap \downarrow q = \text{id}(\vec{x}_k)(\vec{y}) \cap \downarrow q$ for every $\vec{y}$: the point q is in $G\vec{x}_k$, and any refinement of q at which $G\vec{y}$ holds is one at which $\vec{y} = \vec{x}_k$ holds. So $q \in \llbracket \exists! \vec{x} G\vec{x} \rrbracket$, with $\vec{x}_k$ as witness.

So $x \in \llbracket \exists G(\Box \forall \vec{x} (G\vec{x} \rightarrow F\vec{x}) \wedge \Box \exists! \vec{x} G\vec{x}) \rrbracket$, with G as witness. Since x and F were arbitrary, every refinement of any point at which the antecedent of $\text{IC}_{\vec{\sigma}}$ holds of F is one at which its consequent does; so $\text{IC}_{\vec{\sigma}}$ holds at every point, and is valid. $\Box \text{IC}$ is valid by Corollary 3.7. $\square$

The choice of witnesses at the worlds is the usual intraworld choice; the choice in the cloud is the mixing argument familiar from Boolean-valued models. Both use the axiom of choice in the metatheory.

3.6 The Barcan formula

Let BF_{σ} be $\forall F^{\sigma \rightarrow t} (\forall x \Box Fx \rightarrow \Box \forall x Fx)$.

Definition 3.15. $\mathcal{M}$ has no new entities of type σ if for every $x \in S$ and every $a \in \mathbf{D}_x^{\sigma}$ there is $a' \in \mathbf{D}_r^{\sigma}$ with $a \sim_x a'$.

Proposition 3.16. If $\mathcal{M}$ has no new entities of type σ , then BF_{σ} , and hence $\Box \text{BF}_{\sigma}$, is valid in $\mathcal{M}$.

Proof. Let g be an x -assignment, $f = \llbracket F \rrbracket_g$, and let $x' \sqsubseteq x$ be in $\llbracket \forall y \Box Fy \rrbracket_g$. Let $v \in R[x']$, $v' \sqsubseteq v$, and $a \in \mathbf{D}_{v'}^{\sigma}$; we must show $v' \in f(a)$. Choose $a' \in \mathbf{D}_r^{\sigma}$ with $a \sim_{v'} a'$. Since rRx' , a' exists at x' , so $x' \in \Box_R f(a')$, and as $v' \in R[x']$, $v' \in f(a')$. Since f exists at v' , $f(a) \sim_{v'} f(a')$, so $v' \in f(a)$. Hence $x' \in \llbracket \Box \forall y Fy \rrbracket_g$. $\square$

Since Classicism proves that Rigid Comprehension implies Boolean Completeness and Actuality (Bacon and Dorr, 2024), Theorem 3.12 and Corollary 3.7 also give:

Corollary 3.17. In every full model over a world–cloud frame, $\Box$ Boolean Completeness and $\Box$ Actuality are valid.

4 A model of $C.\Box RC.\Box IC.\Box BF$ without Atomicity

We now put the pieces together. The models of this section are full models over rooted sums of forcing frames, with a constant domain of individuals.

Definition 4.1 (Cloud models). Let $\mathfrak{B} = (B_i)_{i \in I}$ be a non-empty family of complete Boolean algebras, each with $0 \neq 1$, and let D be a non-empty set. For each i , let $\mathcal{P}_i$ be the forcing frame on $B_i \setminus \{0\}$ (Example 1.10), and let $\mathcal{F}_{\mathfrak{B}} = \bigoplus_{i \in I} \mathcal{P}_i$ be their rooted sum, with root r (Definition 1.11). Let $\mathbf{D}^e$ be the modalized domain with underlying set D and $a \sim_x b$ iff $a = b$, for every point x . The *cloud model* $\mathcal{M}(\mathfrak{B}, D)$ is the full model over $\langle \mathcal{F}_{\mathfrak{B}}, r \rangle$ determined by $\mathbf{D}^e$ (Proposition 2.11).

So $\mathcal{M}(\mathfrak{B}, D)$ has a single world, the root, which sees every point; below it lie the refinement orders of the algebras B_i , each point of which sees only its own refinements. By Proposition 1.12 and Example 1.10,

$$\text{RO}(\mathcal{F}_{\mathfrak{B}}) \cong \mathbf{2} \times \prod_{i \in I} B_i,$$

the first coordinate recording truth at the root. Every individual exists everywhere, and distinct individuals are distinct everywhere. A proposition is true iff it contains the root. The root is $\sqsubseteq$ -minimal, so truth is bivalent (Lemma 2.13). Since the root sees everything, identity at the root is literal for propositions: $X \sim_r Y$ iff $X = Y$.

Lemma 4.2. $\langle \mathcal{F}_{\mathfrak{B}}, r \rangle$ is a world–cloud frame, with $W = \{r\}$ and cloud $T = \bigcup_i (B_i \setminus \{0\})$.

Proof. The root is $\sqsubseteq$ -isolated, and every other point is transparent, by the definitions of the rooted sum and of forcing frames. $\square$

The heart of the verification of the Barcan formula is that nothing exists in the cloud that is not, up to identity there, already actual. The idea is that an entity existing at a cloud point x is constrained only below x , and can be extended to an entity that exists everywhere by cutting it down to $\langle x \rangle$.

Lemma 4.3 (No new entities). $\mathcal{M}(\mathfrak{B}, D)$ has no new entities of any type.

Proof. Let x be a point and $a \in \mathbf{D}_x^\sigma$; we find $a' \in \mathbf{D}_r^\sigma$ with $a \sim_x a'$. If $x = r$ take $a' = a$. Let x be a cloud point. At type e every individual exists at r , and at type t every proposition exists at r , since the model is full; so take $a' = a$. Let $\sigma = \sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow t$ with $n \geq 1$, and write f for a . Define f' by

$$f'(z_1) \cdots (z_n) := f(z_1) \cdots (z_n) \cap \langle x \rangle.$$

We first show:

(†) if $q \in \langle x \rangle$ and $\vec{a} \sim_q \vec{b}$, then $q \in f\vec{a}$ iff $q \in f\vec{b}$.

Suppose $q \in f\vec{a}$, and let $q' \sqsubseteq q$. Since q' is compatible with x , it has a refinement $q'' \sqsubseteq x$. Then $q'' \in f\vec{a}$ by persistence, $\vec{a} \sim_{q''} \vec{b}$ by persistence of $\sim$, and, since $q'' \in R[x]$ and f exists at x , $f\vec{a} \sim_{q''} f\vec{b}$; so $q'' \in f\vec{b}$. By refinability, $q \in f\vec{b}$. The converse is symmetric.

f' exists at r . By the unfolding (*) in the proof of Proposition 2.11, we must show that if y is any point and $\vec{a} \sim_y \vec{b}$, then $f'\vec{a}$ and $f'\vec{b}$ agree on $R[y]$. Let $q \in R[y]$. Then $\vec{a} \sim_q \vec{b}$ by persistence; if $q \notin \langle x \rangle$, neither $f'\vec{a}$ nor $f'\vec{b}$ contains q , and if $q \in \langle x \rangle$, (†) shows that $q \in f'\vec{a}$ iff $q \in f'\vec{b}$.

$f \sim_x f'$. Let $y \in R[x] = \downarrow x$ and $\vec{a} \sim_y \vec{b}$. Since $\downarrow y \subseteq \downarrow x \subseteq \langle x \rangle$, $f'\vec{b}$ agrees with $f\vec{b}$ on $\downarrow y$, which agrees with $f\vec{a}$ on $\downarrow y = R[y]$, as f exists at x . So $f\vec{a} \sim_y f'\vec{b}$. Unfolding [Definition 2.2](#) as in (*), this gives $f \sim_x f'$. $\square$

Theorem 4.4. *Let $\mathfrak{B}$ and D be as in [Definition 4.1](#). Then:*

- (a) *every theorem of Classicism is valid in $\mathcal{M}(\mathfrak{B}, D)$;*
- (b) $\Box RC$, $\Box IC$ and $\Box BF$ *are valid in $\mathcal{M}(\mathfrak{B}, D)$, at every relational type (and, for BF , at every type);*
- (c) *ND fails at the root, at type t ;*
- (d) *if some B_i is atomless, Atomicity fails at the root, at type t .*

Proof. (a) is [Theorem 3.4](#). (b) follows from [Lemma 4.2](#) and [Theorems 3.12](#) and [3.14](#), and from [Lemma 4.3](#) and [Proposition 3.16](#).

(c) Let $p = S$ and $q = S \setminus \{r\}$; both are propositions by [Proposition 1.12](#). They are distinct, so by [Lemma 2.13](#) the root is in $\llbracket p \neq q \rrbracket$, where p and q are assigned these values. But at every cloud point t , p and q agree on $R[t] = \downarrow t$, so $p \sim_t q$ and $t \notin \llbracket p \neq q \rrbracket$. Hence the root is not in $\llbracket \Box(p \neq q) \rrbracket$, and $\forall p \forall q (p \neq q \rightarrow \Box(p \neq q))$ fails at the root.

(d) Recall from [Bacon and Dorr \(2024\)](#) that Atomicity at type t is

$$\forall p (p \leq \neg p \vee \exists q (\text{Atom } q \wedge q \leq p)), \quad \text{where} \quad \text{Atom} := \lambda q. \forall z ((z \leq q \wedge z \neq q) \leftrightarrow z \leq \neg z).$$

At the root, identity of propositions is literal and $z \leq q$ holds iff $z \subseteq q$, since $R[r] = S$. So, by [Lemma 2.13](#), $\text{Atom } q$ holds at the root iff q is an atom of $\text{RO}(\mathcal{F}_{\mathfrak{B}})$, and $q \leq p$ holds there iff $q \subseteq p$. Let B_i be atomless, and let p be the set of points of the component $\mathcal{P}_i$, a non-empty proposition. Then $p \leq \neg p$ fails at the root. Any atom $q \subseteq p$ of $\text{RO}(\mathcal{F}_{\mathfrak{B}})$ would be an atom of $\text{RO}(\mathcal{P}_i) \cong B_i$, and there are none. So Atomicity fails at the root. $\square$

Taking a single atomless algebra—for instance, the completion of the Cohen forcing $\text{Fn}(\omega, 2)$ —we obtain [Theorem A](#):

Corollary 4.5. $C + \Box RC + \Box IC + \Box BF + \neg \text{Atomicity} + \neg ND$ *has a model.*

Remark 4.6. Several further principles are settled in these models.

- (i) $\Box$ Boolean Completeness and $\Box$ Actuality hold, by [Corollary 3.17](#).
- (ii) The principles B and 5 fail: the root's singleton $\{r\}$ is true, but no cloud point sees the root, so $\Diamond\{r\}$ fails throughout the cloud, and $\Box\Diamond\{r\}$ fails at the root.
- (iii) Atomlessness fails, since $\{r\}$ is a possible proposition with no possible proposition strictly below it.
- (iv) Extensionality ($\forall X \forall Y (\forall \vec{z} (X\vec{z} \leftrightarrow Y\vec{z}) \rightarrow X = Y)$) holds at every cloud point, and so is possible. For if t is a cloud point, X and Y exist at t , and X and Y are coextensive at t , then for $y \in \downarrow t$ and $\vec{a} \sim_y \vec{b}$ we have $X\vec{a} \sim_y X\vec{b}$, and $X\vec{b}$ and $Y\vec{b}$ agree on $\downarrow y$; so $X \sim_t Y$. In particular, at every cloud point there are exactly two propositions: $\forall p (p = \top \vee p = \perp)$ holds throughout the cloud.

The model of [Corollary 4.5](#) is a model of Classicism with $\Box$ Rigid Comprehension in which a single possibility, the actual one, is a world, and every other possibility is part of an atomless region in which the box is idle. Rigid Comprehension holds because the identity relations in the cloud are B -valued, and the rigidification of a property R —the check-closure of its actual extension—holds, in the cloud, of exactly the things that are identical to some actual instance of R ; no property can distinguish these from the actual instances. Intensional Choice holds because, in the cloud, choice is by Boolean mixing. And the Barcan formula holds because nothing exists in the cloud that is not identical there to something actual. In the next section we show that this last feature is inessential.

5 A model of $C.\Box RC.\Box IC$ in which Atomicity fails and BF fails at type t

5.1 Why a new idea is needed

In the cloud models of [Section 4](#), the Barcan formula holds because nothing exists in the cloud that is not identical there to something actual. At type t this has nothing to do with fullness: as long as the root is the only world, the propositions of the cloud cannot outrun the actual ones.

Proposition 5.1. *Let $\mathcal{M}$ be any modalized domain model over a world–cloud frame whose only world is the root. Then BF_t is true in $\mathcal{M}$.*

Proof. Let F exist at r , and suppose $r \in \llbracket \forall p \Box Fp \rrbracket$, so that $Fq = S$ for every $q \in \mathbf{D}_r^t$; in particular $FS = F\emptyset = S$, since S and $\emptyset$ are denotations of closed terms. Let x be a cloud point, $x' \sqsubseteq x$, and $q \in \mathbf{D}_{x'}^t$; we show $x' \in Fq$. Every $y \sqsubseteq x'$ has a refinement y' in q or in $\neg q$. In the first case $q \cap \downarrow y' = \downarrow y'$, so $q \sim_{y'} S$ (both exist at y' , and $R[y'] = \downarrow y'$); since F exists at y' , $Fq \sim_{y'} FS = S$, and $y' \in Fq$. The second case is the same, with $\emptyset$ in place of S . By refinability, $x' \in Fq$. So $\llbracket \forall p Fp \rrbracket$ contains every cloud point, and r , so $r \in \llbracket \Box \forall p Fp \rrbracket$. $\square$

So to refute BF_t we need possibilities other than the root at which genuinely new propositions exist, and at which the box is not idle. Such possibilities cannot simply be added to a full model: if a world u has propositions that do not exist at the root, then functional fullness puts into the domain functions that treat these new propositions differently at different worlds seen from u , and identity statements between such functions then denote propositions that do not exist at the root either.¹ We therefore add to the cloud a second component that is not full: a modalized domain version of one of the full action models of [Bacon and Dorr \(2024\)](#). To do so we first describe a general way of combining models.

5.2 Rooted sums of models

The definitions of [Section 2](#) make sense over any S4 possibility frame, rooted or not; only the definition of truth uses the root. We use them in this generality. If $\mathcal{M}$ is a model over

¹For example, take worlds $a_0 Ra_1 Ra_2 R \dots$, with the propositions existing at a_n being those constant on $\{a_m : m > n\}$. Let f be the identity function on propositions, and let g agree with f except that it reverses the truth value at a_1 of every proposition whose truth values at a_1 and a_2 differ. Both exist at a_0 , but $\text{id}(f)(g)$ contains $a_2, a_3, \dots$ and not a_1 , so it does not exist at a_0 .

$\mathcal{F} = \langle S, \sqsubseteq, R \rangle$, let $\mathbf{D}_*^\sigma := \bigcap_{x \in S} \mathbf{D}_x^\sigma$ be the set of entities of type σ existing everywhere in $\mathcal{M}$. If $\mathcal{M}$ has a root r , then $\mathbf{D}_*^\sigma = \mathbf{D}_r^\sigma$.

Definition 5.2 (Rooted sums of models). Let $\mathcal{M}_i$, for $i \in I$, be models over S4 possibility frames $\mathcal{F}_i = \langle S_i, \sqsubseteq_i, R_i \rangle$ with pairwise disjoint sets of points, and let $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ with root r . Let E_0 be a non-empty set, the *actual individuals*, disjoint from the $|\mathbf{D}_i^e|$, with maps $\pi_i: E_0 \rightarrow \mathbf{D}_{i,*}^e$. The *rooted sum* $\bigoplus_i \mathcal{M}_i$ (over E_0 and the π_i) is the structure over $\langle \mathcal{F}, r \rangle$ defined as follows. For each type σ and each i we define a set $|\mathbf{D}^\sigma|$, a partial map $\rho_i^\sigma: |\mathbf{D}^\sigma| \rightarrow |\mathbf{D}_i^\sigma|$, and the relations $\sim_x^\sigma$.

- *Type e.* $|\mathbf{D}^e| = E_0 \cup \bigcup_i |\mathbf{D}_i^e|$; ρ_i^e is π_i on E_0 , the identity on $|\mathbf{D}_i^e|$, and undefined elsewhere.
- *Type t.* $|\mathbf{D}^t|$ is the set of propositions X of $\mathcal{F}$ with $X \cap S_i \in |\mathbf{D}_i^t|$ for every i , and $\rho_i^t(X) = X \cap S_i$.
- *Relational types $\sigma \rightarrow \tau$.* $|\mathbf{D}^{\sigma \rightarrow \tau}|$ is the set of functions $F: |\mathbf{D}^\sigma| \rightarrow |\mathbf{D}^\tau|$ such that for each i there is $f_i \in |\mathbf{D}_i^{\sigma \rightarrow \tau}|$ with $\rho_i^\tau(Fa) = f_i(\rho_i^\sigma a)$ for every a in the domain of ρ_i^σ . Since ρ_i^σ maps its domain onto $|\mathbf{D}_i^\sigma|$ (by induction on types), f_i is unique, and we let $\rho_i^{\sigma \rightarrow \tau}(F) = f_i$.
- *Identity in the components.* For $x \in S_i$: $a \sim_x b$ iff a and b are in the domain of ρ_i and $\rho_i a \sim_x^{\mathcal{M}_i} \rho_i b$.
- *Identity at the root.* At type e : $a \sim_r b$ iff $a = b \in E_0$. At type t : $X \sim_r Y$ iff $X = Y$ and $X \cap S_i \in \mathbf{D}_{i,*}^t$ for every i . At a relational type $\sigma \rightarrow \tau$: $F \sim_r G$ iff $F \sim_r G$ in $\mathbf{D}^\sigma \Rightarrow \mathbf{D}^\tau$ and $\rho_i F, \rho_i G \in \mathbf{D}_{i,*}^\sigma$ for every i .

So an entity of the rooted sum is a family of entities of the components together with an arbitrary *root part*: an individual existing at the root is an actual individual, with a counterpart in each component; a proposition is a truth value at the root together with a proposition of each component; and a relation is given by a relation of each component together with an arbitrary extension at the root. This is the analogue, for modalized domain models, of the coalesced sums of Bacon (2018a) for modalized domains over ordinary Kripke frames (see also Bacon, 2018b). Up to the obvious identification of individuals, the cloud models of Section 4 are rooted sums of full models over forcing frames, with $E_0 = D$ and each π_i the identity.

Lemma 5.3. $\bigoplus_i \mathcal{M}_i$ is a modalized domain model over $\langle \mathcal{F}, r \rangle$. Moreover, for every $x \in S_i$, formula A , and x -assignment g ,

$$x \in \llbracket A \rrbracket_g^\oplus \quad \text{iff} \quad x \in \llbracket A \rrbracket_{\rho_i \circ g}^{\mathcal{M}_i}.$$

Proof sketch. The relations $\sim_x$ are PERs. Persistence within a component is inherited from $\mathcal{M}_i$; for the root, if $a \sim_r b$ then $\rho_i a = \rho_i b$ exists everywhere in $\mathcal{M}_i$, so $a \sim_x b$ for every $x \in S_i$. Refinability is inherited from the components, and is trivial at the root. That $\mathbf{D}^t \leq \mathbf{P}_\mathcal{F}$ and $\mathbf{D}^{\sigma \rightarrow \tau} \leq \mathbf{D}^\sigma \Rightarrow \mathbf{D}^\tau$ holds at component points follows from the corresponding facts about $\mathcal{M}_i$, since for $x \in S_i$ the set $R[x]$ lies within S_i ; at the root it holds by definition.

The logical operations of $\bigoplus_i \mathcal{M}_i$ lie in the underlying sets, with ρ_i -images the logical operations of $\mathcal{M}_i$. For the quantifiers, note that at a point $x \in S_i$ the entities existing at x are exactly the elements of the domain of ρ_i whose ρ_i -images exist at x in $\mathcal{M}_i$, and that ρ_i maps them onto the entities existing at x in $\mathcal{M}_i$. The operations exist at every point of each

component because those of $\mathcal{M}_i$ do, and at the root because their images exist everywhere in each $\mathcal{M}_i$. By induction on terms, $\rho_i \llbracket A \rrbracket_g^\oplus = \llbracket A \rrbracket_{\rho_i \circ g}^{\mathcal{M}_i}$ whenever $\rho_i \circ g$ is defined on the free variables of A ; for an abstraction, the function $d \mapsto \llbracket A \rrbracket_{g[v \mapsto d]}^\oplus$ lies in $|\mathbf{D}^{\sigma \rightarrow \tau}|$, with component $\llbracket \lambda v. A \rrbracket_{\rho_i \circ g}^{\mathcal{M}_i}$, because $\mathcal{M}_i$ is a model. For a formula and an x -assignment this is the displayed equivalence. $\square$

So truth at a point of a component is truth in that component. The root sees all the components. A sentence $\Box A$ is true at the root iff A is true at the root and at every point of every component, and the quantifiers at the root range over the actual individuals, and over families of entities existing everywhere in the components, together with arbitrary root parts.

To transfer Rigid Comprehension and Intensional Choice from the components to the root, we need two conditions on the components. In the following, E ranges over sets of tuples of entities in $\mathbf{D}_{i,*}$ of some fixed types $\vec{\sigma}$.

Definition 5.4. A model $\mathcal{M}_i$ over $\mathcal{F}_i$ is *rigidifiable* if for every such E there is $X \in \mathbf{D}_{i,*}^{\vec{\sigma} \rightarrow t}$ such that

- (i) $X\vec{a} = S_i$ for every $\vec{a} \in E$;
- (ii) $S_i \subseteq \llbracket \forall \vec{z}(X\vec{z} \rightarrow Y\vec{z}) \rrbracket$, whenever $Y \in \mathbf{D}_{i,*}^{\vec{\sigma} \rightarrow t}$ and $Y\vec{a} = S_i$ for every $\vec{a} \in E$;
- (iii) X is rigid at every point of S_i .

$\mathcal{M}_i$ *admits global choice* if for every $F \in \mathbf{D}_{i,*}^{\vec{\sigma} \rightarrow t}$ with $S_i \subseteq \llbracket \exists \vec{x} F\vec{x} \rrbracket$ there is $G \in \mathbf{D}_{i,*}^{\vec{\sigma} \rightarrow t}$ with $S_i \subseteq \llbracket \forall \vec{x}(G\vec{x} \rightarrow F\vec{x}) \rrbracket$ and $S_i \subseteq \llbracket \exists ! \vec{x} G\vec{x} \rrbracket$.

Lemma 5.5.

- (a) A full model over a forcing frame, with a constant domain of individuals, is rigidifiable and admits global choice, and validates RC and IC.
- (b) A model $\mathcal{M}_i$ with a $\sqsubseteq$ -minimal root r_i in which $\Box RC$ and $\Box IC$ are true is rigidifiable and admits global choice, provided that it is extensionally full at its root: for every set E of tuples existing at r_i , closed under $\sim_{r_i}$, there is a relation existing at r_i whose extension at r_i is E .

Proof. (a) The frame is a world–cloud frame without worlds, so RC and IC are valid by the proofs of [Theorems 3.12](#) and [3.14](#) (the case of a cloud point). For rigidifiability, take $X = \check{E}$ ([Definition 3.11](#)). It exists everywhere, by the argument in the proof of [Theorem 3.12](#); it satisfies (i) by persistence; it satisfies (ii) because if $Y\vec{a} = S_i$ and $v \in \text{id}(\vec{a})(\vec{z})$ then $Y\vec{a} \sim_v Y\vec{z}$, so $v \in Y\vec{z}$; hence $X\vec{z} \subseteq Y\vec{z}$ by refinability of $Y\vec{z}$; and it satisfies (iii) by [Lemma 3.10](#). Global choice is the cloud case of the proof of [Theorem 3.14](#), applied to a maximal antichain of decided points in the whole frame.

(b) Since r_i is $\sqsubseteq$ -minimal, truth at r_i is bivalent and its quantifiers range over what exists there ([Lemma 2.13](#)), so true existential sentences have witnesses existing at r_i . Given E , let X be a rigid relation existing at r_i that is coextensive at r_i with a relation whose extension at r_i is the $\sim_{r_i}$ -closure of E ; it exists by extensional fullness and RC at r_i . Since X is persistent at r_i and $R[r_i] = S_i$, (i) holds. Since X is inextensible at r_i , (ii) holds. And (iii) holds since $\text{Rig } X$ begins with a box. For global choice, apply IC at r_i , noting that $S_i \subseteq \llbracket \exists \vec{x} F\vec{x} \rrbracket$ says that $\Box \exists \vec{x} F\vec{x}$ holds at r_i . $\square$

Theorem 5.6 (Transfer). *Let $\mathcal{M} = \oplus_i \mathcal{M}_i$ be a rooted sum of models, with $I \neq \emptyset$.*

- (a) *Classicism is valid in $\mathcal{M}$.*
- (b) *If every $\mathcal{M}_i$ validates RC and is rigidifiable, then $\Box\text{RC}$ is valid in $\mathcal{M}$.*
- (c) *If every $\mathcal{M}_i$ validates IC and admits global choice, then $\Box\text{IC}$ is valid in $\mathcal{M}$.*
- (d) *If some $\mathcal{M}_i$ has a $\sqsubseteq$ -minimal root r_i at which BF_σ fails, σ relational, then BF_σ fails at the root of $\mathcal{M}$.*
- (e) *ND_t fails at the root of $\mathcal{M}$.*
- (f) *If for some i the propositions in $\mathbf{D}_{i,*}^t$ form an atomless algebra under inclusion, then Atomicity fails at the root of $\mathcal{M}$.*

Proof. (a) is [Theorem 3.4](#) and [Lemma 5.3](#). At the root, by [Lemma 2.13](#), quantifiers range over entities existing at the root, and identity is $\sim_r$.

(b) By [Lemma 5.3](#), RC holds at every component point. At the root, let R exist at r , let E_r be its extension there, and for each i let X_i witness rigidifiability of $\mathcal{M}_i$ for $E_i = \{\rho_i \vec{a} : \vec{a} \in E_r\}$. Let X have root extension E_r and components X_i ; it lies in $|\mathbf{D}|$ (its values on arguments outside the domain of ρ_i being arbitrary), and exists at r , since E_r is closed under $\sim_r$ and each X_i exists everywhere in $\mathcal{M}_i$. X is coextensive with R at r . For rigidity, by (iii) and [Lemma 5.3](#) it suffices to check $\psi(Y)$ (as in the proof of [Theorem 3.12](#)) at r for each Y existing at r , with components Y_i . By [Lemmas 2.13](#) and [5.3](#), $\Box \forall \vec{z}(X\vec{z} \rightarrow Y\vec{z})$ holds at r iff E_r is included in the root extension of Y and $S_i \subseteq \llbracket \forall \vec{z}(X_i \vec{z} \rightarrow Y_i \vec{z}) \rrbracket^{\mathcal{M}_i}$ for every i ; and $\forall \vec{z}(X\vec{z} \rightarrow \Box Y\vec{z})$ holds at r iff every $\vec{a} \in E_r$ is in the root extension of Y , and $Y_i \rho_i \vec{a} = S_i$ for every i . The first implies the second by (i), and the second implies the first by (ii). So RC holds at r , and hence is valid; $\Box\text{RC}$ is valid by [Corollary 3.7](#).

(c) IC holds at component points by [Lemma 5.3](#). At the root, let F exist at r with $r \in \llbracket \Box \exists \vec{x} F\vec{x} \rrbracket$. Then the root extension of F contains some $\vec{a}_0$, and $S_i \subseteq \llbracket \exists \vec{x} F_i \vec{x} \rrbracket$ for every i ; let G_i witness global choice for F_i . Let G have as root extension the $\sim_r$ -class of $\vec{a}_0$, which is included in the root extension of F , and components G_i . Then G exists at r , and $\Box \forall \vec{x}(G\vec{x} \rightarrow F\vec{x})$ and $\Box \exists! \vec{x} G\vec{x}$ hold at r , at the root by [Lemma 2.13](#) and at component points by [Lemma 5.3](#).

(d) Since r_i is $\sqsubseteq$ -minimal, [Lemma 2.13](#) gives some F_i existing at r_i , with $\forall x \Box F_i x$ true and $\Box \forall x F_i x$ false at r_i . Let F have component F_i , every other component the constant top function, and root part true of every root argument. Then $\forall x \Box Fx$ holds at r : for each a existing at r , $\rho_i a$ exists at r_i , so $F_i \rho_i a = S_i$. But some point of S_i is not in $\llbracket \forall x Fx \rrbracket$, by [Lemma 5.3](#), so $\Box \forall x Fx$ fails at r .

(e) S and $S \setminus \{r\}$ both exist at r and are distinct, but at every component point they are identical, since their components coincide.

(f) At the root, identity of propositions is literal, and $z \leq q$ holds iff $z \subseteq q$, so $\text{Atom } q$ holds at r iff q is an atom of the algebra $\mathbf{D}_r^t$ of propositions existing at r . Let p have component S_i in $\mathcal{M}_i$, and $\emptyset$ elsewhere. If $q \subseteq p$ is non-empty and exists at r , then $q \cap S_i$ is a non-zero element of the atomless algebra $\mathbf{D}_{i,*}^t$, so has a non-zero element strictly below it, which (with empty components elsewhere and false at the root) is a proposition existing at r strictly below q . So no atom lies below p , and Atomicity fails at r . $\square$

5.3 A component in which BF fails at type t

We now describe the second component, using the action models of [Bacon and Dorr \(2024, §3\)](#). Let $\mathcal{K}$ be the monoid with two elements 1 and k , with $k \circ k = k$, viewed as a category with one object W_0 . Let $\mathbf{A}_k$ be the full action model over $\mathcal{K}$ with a single individual ([Bacon and Dorr, 2024, §3.5](#)). Its propositions are the four subsets of $\{1, k\}$. The arrow k acts on them by $k^t(p) = \{i : i \circ k \in p\}$, which is $\{1, k\}$ if $k \in p$ and $\emptyset$ otherwise. So k^t is not surjective: moving along k , every proposition becomes the top or the bottom one, yet one arrives at an object with four propositions again.

Lemma 5.7. *The following hold in $\mathbf{A}_k$, at W_0 :*

- (a) $\Box RC$ is true, and every subset of a domain of W_0 is the extension of some relation;
- (b) $\Box IC$ is true;
- (c) BF_t is false.

Proof. (a) [Bacon and Dorr \(2024, §3.5\)](#) show that RC and $\Box RC$ are true in every intensionally full action model, using exactly the relations α_X that witness the second claim. (c) $\mathbf{A}_k$ is quasi-functionally full and k^t is not surjective, so BF_t fails by [Bacon and Dorr \(2024, §3.5\)](#), who show that in a quasi-functionally full action model BF_σ holds only if every h^σ with source W_0 is surjective. (b) We give the argument at W_0 ; the argument at other arrows is the same. Let $F \in W_0^{\vec{\sigma} \rightarrow t}$, with $\Box \exists \vec{x} F \vec{x}$ true at W_0 . For each arrow $h: W_0 \rightarrow W_0$ choose $\vec{b}_h \in W_0^{\vec{\sigma}}$ with $\text{id} \in F \langle h, \vec{b}_h \rangle$. Define

$$G \langle h, \vec{a} \rangle := \{i : i^{\vec{\sigma}} \vec{a} = \vec{b}_{i \circ h}\}.$$

G is natural: $j^t G \langle h, \vec{a} \rangle = \{i : (i \circ j)^{\vec{\sigma}} \vec{a} = \vec{b}_{i \circ j \circ h}\} = G \langle j \circ h, j^{\vec{\sigma}} \vec{a} \rangle$; so $G \in W_0^{\vec{\sigma} \rightarrow t}$ by fullness. If $i \in G \langle h, \vec{a} \rangle$ then $i^t F \langle h, \vec{a} \rangle = F \langle i \circ h, i^{\vec{\sigma}} \vec{a} \rangle = F \langle i \circ h, \vec{b}_{i \circ h} \rangle \ni \text{id}$, so $i \in F \langle h, \vec{a} \rangle$; hence $\Box \forall \vec{x} (G \vec{x} \rightarrow F \vec{x})$. And $\text{id} \in G \langle h, \vec{a} \rangle$ iff $\vec{a} = \vec{b}_h$, so $\exists! \vec{x} G \vec{x}$ holds at every h , and $\Box \exists! \vec{x} G \vec{x}$ is true. $\square$

[Bacon and Dorr \(2024, §3.5\)](#) show that every action model can be *unravelled* into an action model on a preorder category in which the same formulas are true, and that action models on preorder categories correspond to modalized domain models over Kripke frames, whose entities are the “modal worms” of the action model. Applying this to $\mathbf{A}_k$ yields a modalized domain model $\mathcal{M}_k$ over a rooted Kripke frame, whose entities existing at the root correspond to the elements of the domains of $\mathbf{A}_k$ at W_0 . Intuitively, the frame is a chain $a_0 R a_1 R a_2 R \dots$, corresponding to the paths $1, k, kk, \dots$, and the propositions existing at a_n are those whose truth values at $a_{n+1}, a_{n+2}, \dots$ coincide: following k makes every old proposition settled, while new propositions arise about the possibility one has arrived at. By [Lemma 5.7](#), $\Box RC$ and $\Box IC$ are true in $\mathcal{M}_k$, $\mathcal{M}_k$ is extensionally full at its root, and BF_t fails at its root. Since the frame is a Kripke frame, its root is $\sqsubseteq$ -minimal and every point of it is visible from the root, so RC and IC are valid in $\mathcal{M}_k$. By [Lemma 5.5\(b\)](#), $\mathcal{M}_k$ is rigidifiable and admits global choice.

5.4 The model

Let $\mathcal{M}_C$ be the full model over the forcing frame of the Cohen algebra (or any complete atomless Boolean algebra), with domain of individuals $D = \omega$, and let $\mathcal{M}_k$ be as above. Let

$\mathcal{M}_{C,k} = \mathcal{M}_C \oplus \mathcal{M}_k$, with actual individuals $E_0 = \omega$, π_C the identity and π_k the constant map to the single individual of $\mathcal{M}_k$.

Theorem 5.8. *In $\mathcal{M}_{C,k}$, every theorem of Classicism is valid, $\Box RC$ and $\Box IC$ are valid, and Atomicity, ND and BF_t fail at the root.*

Proof. By Theorem 5.6: (a)–(c) using Lemma 5.5, (d) using Lemma 5.7(c), (e), and (f) using the atomless algebra $RO(\mathcal{M}_C)$. $\square$

This proves Theorem B. In $\mathcal{M}_{C,k}$ the actual world sees two quite different regions. In the cloud, propositions are as coarse as possible: there are only two of them from the inside. In the chain, by contrast, new propositions keep coming into existence: at each a_n there are propositions that settle a question about a_n itself which no earlier possibility could even raise. The atomless cloud refutes Atomicity, and the chain refutes BF_t .

6 Models of Mathematical Possibilism of arbitrarily large cardinality

6.1 Mathematical Possibilism

Mathematical Possibilism says, roughly, that every small mathematical structure of a certain kind is realized in logical space as the algebra of propositions entailing some proposition (Bacon, 2026). We state it for complete Boolean algebras. For X of type $\sigma \rightarrow t$ and $\preceq$ of type $\sigma \rightarrow \sigma \rightarrow t$, $CBA(X, \preceq)$ is the higher-order formula saying that $\preceq$ partially orders the X s into a complete Boolean algebra, where completeness says that for every Z of type $\sigma \rightarrow t$ the X s that are Z have a $\preceq$ -least upper bound among the X s. $(X, \preceq) \cong (Y, \preceq')$ says that some relation is a bijection between the X s and the Y s that preserves and reflects the orders. $X \preceq Y$ says that some relation is a one-one function from the X s into the Y s, and $X \prec Y$ is $X \preceq Y \wedge \neg Y \preceq X$. Fin is any of the standard higher-order definitions of finiteness (inductive, or Dedekind). Following Bacon (2026), a complete Boolean algebra is *small* if it is finite or has fewer elements than there are propositions:

$$\text{SmallCBA}(X, \preceq) := CBA(X, \preceq) \wedge (\text{Fin } X \vee X \prec \lambda p^t. \top).$$

Mathematical Possibilism (MP) is the schema, with one instance for each type σ ,

$$\forall X^{\sigma \rightarrow t} \forall \preceq^{\sigma \rightarrow \sigma \rightarrow t} \left(\text{SmallCBA}(X, \preceq) \rightarrow \exists p^t (p \neq \top \wedge (X, \preceq) \cong (\lambda q. q \leq p, \leq)) \right).$$

Here $(\lambda q. q \leq p, \leq)$ is the algebra of propositions entailing p , ordered by entailment. The restriction to small algebras is needed to avoid Cantorian paradox.

6.2 Cardinal arithmetic

Throughout this section, $\mu > \omega$ is a *strong limit* cardinal ($2^\nu < \mu$ for all $\nu < \mu$) of countable cofinality.

Lemma 6.1. (a) *Above every cardinal λ there is such a μ .* (b) $\mu^{\aleph_0} = 2^\mu$.

Proof. (a) Let $\lambda_0 = \max(\lambda, \omega)$, $\lambda_{n+1} = 2^{\lambda_n}$, and $\mu = \sup_n \lambda_n$. (b) Write $\mu = \sup_n \mu_n$ with $\mu_n < \mu$ increasing. Then $2^\mu = \prod_n 2^{\mu_n} \leq \prod_n \mu = \mu^{\aleph_0} \leq (2^\mu)^{\aleph_0} = 2^\mu$. $\square$

Theorem 6.2 (Pierce, 1958; see also Comfort and Hager, 1972). *Every infinite complete Boolean algebra C satisfies $|C|^{\aleph_0} = |C|$.*

Corollary 6.3. *No complete Boolean algebra has cardinality in $[\mu, 2^\mu)$. Hence every complete Boolean algebra with fewer than 2^μ elements has fewer than μ elements.*

Proof. If $\mu \leq \nu < 2^\mu$ then $\nu^{\aleph_0} \geq \mu^{\aleph_0} = 2^\mu > \nu$, so no infinite complete Boolean algebra has cardinality ν . $\square$

Definition 6.4 (The lottery family). Let $\mathfrak{B}_\mu = (B_i)_{i \in I_\mu}$ contain exactly one representative of each isomorphism class of complete Boolean algebras C with $2 \leq |C| < \mu$.

The product $\prod_i B_i$ is the completion of the lottery sum of the B_i , as a forcing notion; the corresponding frame offers, side by side, a copy of the refinement order of every complete Boolean algebra of cardinality less than μ .

Lemma 6.5. $|I_\mu| \leq \mu$ and $|\prod_i B_i| = 2^\mu$.

Proof. A complete Boolean algebra of cardinality $\nu < \mu$ is isomorphic to one with universe ν , so there are at most $2^{\nu \times \nu} < \mu$ isomorphism types of cardinality ν , and $|I_\mu| \leq \sum_{\nu < \mu} 2^\nu = \mu$. Hence $|\prod_i B_i| \leq \mu^\mu = 2^\mu$. Conversely, with μ_n strictly increasing as in Lemma 6.1, the algebras $\mathcal{P}(\mu_n)$ are pairwise non-isomorphic with $2^{\mu_n} < \mu$ elements, so they occur among the B_i , and $|\prod_i B_i| \geq \prod_n 2^{\mu_n} = 2^\mu$. $\square$

6.3 The models

Let $\mathcal{M}_\mu = \mathcal{M}(\mathfrak{B}_\mu, \omega)$ be the cloud model of Definition 4.1 for the lottery family, with a countably infinite domain of individuals. By Proposition 1.12 and Lemma 6.5, $\text{RO}(\mathcal{F}_{\mathfrak{B}_\mu}) \cong 2 \times \prod_i B_i$ has 2^μ elements, all of which exist at the root, and identity of propositions at the root is literal; so there are 2^μ propositions at the root.

To evaluate MP at the root we need to know that the root computes cardinalities and Boolean algebras correctly. At the root, quantifiers range over the entities existing there, and identity is $\sim_r$ (Lemma 2.13). The following lemma shows that relation variables range, through their extensions at the root, over *all* relations on these domains.

Lemma 6.6 (Extensional fullness at the root). *In any cloud model, for every relational type $\vec{\sigma} \rightarrow t$ and every set E of tuples existing at r that is closed under $\sim_r$, there is a relation existing at r whose extension at r is E .*

Proof. Let $F\vec{z}$ be $\{r\}$ if $\vec{z} \in E$ and $\emptyset$ otherwise. On the cloud, F is constantly $\emptyset$, so it respects identity at every cloud point; at the root, $\sim_r$ -related arguments are both in E or both outside it. So F exists at r , by the unfolding (*) in the proof of Proposition 2.11. $\square$

Lemma 6.7. *Let X be of type $\sigma \rightarrow t$, Y of type $\tau \rightarrow t$, $\preceq$ of type $\sigma \rightarrow \sigma \rightarrow t$, and p a proposition, all existing at the root of a cloud model. Write X_r for the set of $\sim_r$ -classes of entities existing at r to which X applies at r , and similarly for the others. Then, at the root:*

- (a) $\text{CBA}(X, \preceq)$ holds iff $(X_r, \preceq_r)$ is a complete Boolean algebra;
- (b) $X \preceq Y$ holds iff $|X_r| \leq |Y_r|$, and $X \prec Y$ iff $|X_r| < |Y_r|$; and if $\text{Fin } X$ holds then X_r is finite;
- (c) $(X, \preceq) \cong (\lambda q. q \leq p, \leq)$ holds iff $(X_r, \preceq_r)$ is isomorphic to $\{q \in \text{RO}(\mathcal{F}) : q \subseteq p\}$ ordered by inclusion; and $p \neq \top$ holds iff $p \neq S$.

Proof. By Lemma 2.13, the formulas in question hold at the root iff they hold in the Tarskian structure whose domain of type σ is $\mathbf{D}_r^\sigma / \sim_r$, with each relation interpreted by its extension at the root. By Lemma 6.6, the relation variables of each type range, via their extensions, over all relations on these domains. So completeness, one–one correspondence and isomorphism are computed as in the metatheory, which gives (a), (b) and the first part of (c); for inductive finiteness, the property of properties whose root extension consists of the properties with finite root extensions shows that $\text{Fin } X$ implies that X_r is finite, and Dedekind finiteness is similar. For (c), $q \leq p$ holds at the root iff $q \subseteq p$, since $R[r] = S$; and the root extension of $\lambda p^t. \top$ is $\text{RO}(\mathcal{F})$, since identity of propositions at the root is literal. $\square$

Theorem 6.8. *MP is true in $\mathcal{M}_\mu$.*

Proof. Let X and $\preceq$ exist at r , with $\text{SmallCBA}(X, \preceq)$ true at r . By Lemma 6.7, $C = (X_r, \preceq_r)$ is a complete Boolean algebra that is either finite or has fewer than $|\text{RO}(\mathcal{F})| = 2^\mu$ elements. Either way $|C| < 2^\mu$, and so $|C| < \mu$ by Corollary 6.3. If $|C| = 1$, let $p = \emptyset$. Otherwise $C \cong B_i$ for some $i \in I_\mu$; let p be the set of points of the component of B_i . In either case $p \neq S$, and the propositions included in p form an algebra isomorphic to C , by Proposition 1.12 and Example 1.10. Fix an isomorphism h . By Lemma 6.6 there is a relation existing at r whose extension at r is the graph of h , which witnesses $(X, \preceq) \cong (\lambda q. q \leq p, \leq)$ at r by Lemma 6.7. So the instance of MP holds at r . $\square$

Corollary 6.9. *For every cardinal λ there is a model of $\text{C} + \square\text{RC} + \square\text{IC} + \square\text{BF} + \text{MP}$ in which there are more than λ propositions. In each such model there are more propositions than rigid properties of individuals.*

Proof. Choose $\mu > \lambda$ by Lemma 6.1, and apply Theorems 4.4 and 6.8. For the second claim: by Proposition 3.13, a property of individuals rigid at the root is identical there to the check-closure of its root extension, a subset of ω , so up to identity there are $2^{\aleph_0}$ such properties at the root; since $2^{\aleph_0} < 2^\mu$, Lemma 6.7 shows that the sentence saying there are fewer rigid properties of individuals than propositions is true. $\square$

This proves Theorem C, and with it the conjecture of Bacon (2026). Since the lottery family contains atomless algebras, Atomicity fails in $\mathcal{M}_\mu$, as it must: Bacon (2026) shows that $\square\text{RC}$, Intensional Choice and MP jointly entail the structural contingency of the continuum hypothesis.

Remark 6.10 (Why countable cofinality). The choice of μ is essential. The relative algebras of $\text{RO}(\mathcal{F}_{\mathfrak{B}_\mu})$ are, up to an extra atom, products of relative algebras of the B_i , each of cardinality less than μ . If the cut-off κ had uncountable cofinality—for instance, if κ were inaccessible—then $\kappa^{\aleph_0} = \kappa$, and the Cohen algebra adding κ reals would have cardinality $\kappa < 2^\kappa$ and so be small. But it is homogeneous and atomless, so it is not isomorphic to any such product: each

non-zero factor would be a relative algebra of it, hence isomorphic to it, hence of cardinality κ . Enlarging the family does not help, since it raises the number of propositions along with it. For the same reason, MP fails in every $\mathcal{M}_\mu$ if ‘small’ is read as ‘no bigger than the number of propositions’: the Cohen algebra adding 2^μ reals has exactly 2^μ elements.

Remark 6.11 (MP is true but not necessary). Inside the cloud there are only two propositions (Remark 4.6), so the four-element Boolean algebra, which is small, is realized at no cloud point. So $\mathcal{M}_\mu$ makes MP true but possibly false. Likewise, it is true but contingent that there are more than $2^{\aleph_0}$ propositions. This is harmless on the convention, followed throughout, that added principles are not automatically necessitated. But it is a feature of the architecture, since the idleness of the box in the cloud is exactly what makes Rigid Comprehension hold there.

7 Logical atomism without Atomicity

7.1 The Tractarian principles

We follow Bacon (MS). The language is extended with a non-logical constant E of type $t \rightarrow t$, read ‘is elementary’. For relations V of type $t \rightarrow t \rightarrow t$ we write $V[p] = q$ for Vpq . Let GLB be the greatest-lower-bound relation on propositions, defined from $\leq$ as in Bacon and Dorr (2024).

- A *truth-value assignment* is a rigid V_0 of type $t \rightarrow t \rightarrow t$ that is functional, relates exactly the elementary propositions to something, and relates them only to $\top$ or $\perp$.
- A *valuation* is a rigid V of type $t \rightarrow t \rightarrow t$ that is functional, relates every proposition to something, relates propositions only to $\top$ or $\perp$, and preserves greatest lower bounds and negation: for every property X of propositions, $V[p] = \top$ for every p such that Xp iff $V[q] = \top$ for the greatest lower bound q of the X s; and $V[p] = q$ iff $V[\neg p] = \neg q$.
- V *extends* V_0 if they agree on the elementary propositions.
- The *world according to* V_0 is $W_{V_0} := \forall p(V_0[p] = \top \rightarrow p) \wedge \forall p(V_0[p] = \perp \rightarrow \neg p)$.
- A *world proposition* is a possible proposition w such that, for every p , $w \leq p$ or $w \leq \neg p$.

The Tractarian Truth-Table Postulate (TTTP) Necessarily, every truth-value assignment extends to a unique valuation.

Aletheic Supervenience (AS) Necessarily, for every truth-value assignment V_0 , W_{V_0} is a world proposition.

The existence half of TTTP says that every way of assigning truth values to the elementary propositions extends to a consistent assignment of truth values to all propositions; the uniqueness half says that the truth values of the elementary propositions determine those of all propositions. AS says that every assignment of truth values to the elementary propositions determines a maximally specific possibility. Bacon (MS) conjectures that both are consistent with Classicism, $\Box RC$ and the failure of Atomicity. If there are only finitely many elementary propositions, the complete descriptions in terms of them form a finite partition of $\top$, so AS implies Atomicity; the models below therefore have infinitely many elementary propositions.

7.2 The model

Let Λ be an infinite set, the *elementary states*, and $f_*: \Lambda \rightarrow \mathbf{2}$ a fixed function, the *actual assignment*.

Definition 7.1 (The atomist frame). Let $\text{Fn}(\Lambda, 2)$ be the set of finite partial functions from Λ to 2 , ordered by reverse inclusion (so $p' \sqsubseteq p$ iff $p' \supseteq p$). The *atomist frame* $\mathcal{F}_\Lambda$ has as points

- a *world* w_f for each $f: \Lambda \rightarrow \mathbf{2}$, with root $r = w_{f_*}$;
- the elements of $\text{Fn}(\Lambda, 2)$, forming the *cloud*.

Refinement is reverse inclusion on the cloud; worlds are refined only by themselves. Every world sees every point, and each cloud point sees exactly its refinements.

So the worlds form a cluster, each seeing all the others and the whole cloud, while the cloud is the Cohen forcing for adding a function from Λ to 2 . This is a world–cloud frame; the three frame conditions, reflexivity and transitivity are immediate. By [Lemma 3.9](#),

$$\text{RO}(\mathcal{F}_\Lambda) \cong \mathcal{P}(2^\Lambda) \times C_\Lambda,$$

where C_Λ is the Cohen algebra $\text{RO}(\text{Fn}(\Lambda, 2))$, which is atomless.

For $\lambda \in \Lambda$, let

$$\hat{\lambda} := \{w_f : f(\lambda) = 1\} \cup \{p \in \text{Fn}(\Lambda, 2) : p(\lambda) = 1\},$$

the proposition that the state λ obtains. In the cloud, $\hat{\lambda}$ is the λ -th Cohen bit c_λ ; at the worlds, it is the canonical generator of the double powerset algebra $\mathcal{P}(2^\Lambda)$. So the elementary propositions $\{\hat{\lambda} : \lambda \in \Lambda\}$ are exactly the completely independent generators of $\text{RO}(\mathcal{F}_\Lambda) \cong C_\Lambda \times \mathcal{PP}(\Lambda)$ given by [Fritz and Bacon \(MS, Lemma 4.6\)](#).

Definition 7.2. The *atomist model* $\mathcal{M}_\Lambda$ is the full model over $\langle \mathcal{F}_\Lambda, r \rangle$ with a non-empty constant domain D of individuals (identity literal everywhere), in which E is interpreted by

$$I(E)(X) := \begin{cases} \{w_f : f \in 2^\Lambda\} & \text{if } X = \hat{\lambda} \text{ for some } \lambda \in \Lambda, \\ \emptyset & \text{otherwise.} \end{cases}$$

So the elementary propositions are elementary at every world and at no cloud point. $I(E)$ exists everywhere: identity of propositions at a world is literal, since worlds see everything, and on the cloud $I(E)$ is constantly empty. So $\mathcal{M}_\Lambda$ is a model by [Proposition 2.11](#).

7.3 Verification

We first record what rigidity amounts to at a world of this model.

Lemma 7.3. *Let X exist at a world w of $\mathcal{M}_\Lambda$ and be rigid there, with extension E at w . Then X has extension E at every world, and $X \sim_w \check{E}$. In particular, two relations that are rigid at w and coextensive at w are identical at w .*

Proof. By [Proposition 3.13](#), $X \sim_w \check{E}$. At any world v , which is $\sqsubseteq$ -minimal and sees w and is seen by it, $\check{E}$ holds of $\bar{z}$ iff $\bar{z} \sim_v \bar{y}$ for some $\bar{y} \in E$, iff $\bar{z} \in E$, since identity at worlds coincides with identity at w (both being the function-space relation over the whole frame). $\square$

Theorem 7.4. *In $\mathcal{M}_\Lambda$: every theorem of Classicism is valid; $\Box RC$, $\Box IC$ and $\Box BF$ are valid; Atomicity and ND fail at the root; and TTTP, AS, $\Box TTTP$ and $\Box AS$ are true.*

Proof. Classicism, $\Box RC$ and $\Box IC$ follow from [Theorems 3.4](#), [3.12](#) and [3.14](#). For $\Box BF$, the proof of [Lemma 4.3](#) applies verbatim: every world sees every point, so an entity existing at a world exists at the root, and an entity existing at a cloud point x is identical there to its cut-down to $\langle x \rangle$, which exists at the root; then apply [Proposition 3.16](#). Atomicity fails at the root because the cloud is a non-empty proposition with no atom below it, as in [Theorem 4.4\(d\)](#), and ND fails because S and the set of cloud points are distinct at the root but identical throughout the cloud.

It remains to show that the matrices of TTTP and AS—the formulas within the initial box—hold at every point; then TTTP and AS hold at every point, and so do their necessitations.

At a world w . By [Lemma 3.3](#), the formulas are evaluated classically at w , with quantifiers over the entities existing at w , and $\leq$ at w is inclusion, since $R[w] = S$. So:

- The elementary propositions at w are the $\hat{\lambda}$, which are pairwise distinct.
- A truth-value assignment V_0 at w has, by [Lemma 7.3](#), the same extension at every world: the graph of the function $\hat{\lambda} \mapsto f(\lambda)$ (identifying 1 with S and 0 with $\emptyset$), for some $f: \Lambda \rightarrow 2$. Conversely, the check-closure of any such graph is a truth-value assignment at w .
- $V_0[\hat{\lambda}] = S$ holds at w for $f(\lambda) = 1$, so by persistence of V_0 it holds everywhere; so $W_{V_0} \subseteq \hat{\lambda}$ for $f(\lambda) = 1$, and likewise $W_{V_0} \subseteq \neg \hat{\lambda}$ for $f(\lambda) = 0$. In the cloud, $\bigcap_{f(\lambda)=1} c_\lambda \cap \bigcap_{f(\lambda)=0} \neg c_\lambda = \emptyset$, since Λ is infinite and no finite condition decides infinitely many bits; at the worlds, the intersection is $\{w_f\}$. So $W_{V_0} \subseteq \{w_f\}$. Conversely $w_f \in W_{V_0}$, since by [Lemma 3.3](#) it suffices that $w_f \in p$ whenever $V_0[p] = \top$ holds at w_f , and by [Lemma 7.3](#) that happens only for $p = \hat{\lambda}$ with $f(\lambda) = 1$; similarly for the second conjunct. So $W_{V_0} = \{w_f\}$, which is an atom of $RO(\mathcal{F}_\Lambda)$; it is possible at w , and for every proposition p either $\{w_f\} \subseteq p$ or $\{w_f\} \subseteq \neg p$. So W_{V_0} is a world proposition at w , and AS holds at w .
- A valuation V at w has a graph that is the graph of a function $h: RO(\mathcal{F}_\Lambda) \rightarrow \{S, \emptyset\}$ preserving arbitrary meets (by [Lemma 6.6](#), every set of propositions is the extension at w of some property) and complements, that is, a complete homomorphism to 2 . Such homomorphisms correspond to the atoms of $RO(\mathcal{F}_\Lambda)$, which are the singletons $\{w_g\}$, since the cloud is atomless. V extends V_0 iff $h(\hat{\lambda}) = f(\lambda)$ for all λ , iff the corresponding atom is $\{w_f\}$. So there is exactly one possible extension at w of each truth-value assignment. A valuation with that extension exists, namely the check-closure of its graph, which is rigid and satisfies the remaining conditions at w ; and any two such are identical at w , by [Lemma 7.3](#). So TTTP holds at w .

([Lemma 6.6](#) was stated for cloud models, but its proof applies here, with the set of all worlds in place of $\{r\}$, since every world sees every point and identity is the same at every world.)

At a cloud point. Let T be the theory consisting of Classicism together with the sentences

$$\forall p (p = \top \vee p = \perp), \quad \text{Extensionality}, \quad \forall p \neg Ep.$$

Each of these holds at every cloud point: the first two by [Remark 4.6](#), whose argument uses only the transparency of cloud points, and the third because $I(E)$ is empty on the cloud. Since the cloud is closed under R , and the theorems of Classicism hold at every point, every sentence derivable in Classicism from these sentences by modus ponens holds at every cloud point. And the matrices of TTTP and AS are so derivable. In the presence of the first sentence, $\Box p \leftrightarrow p$, so rigidity is trivial. The only truth-value assignments relate nothing to anything, since nothing is elementary. Their world is $\top$, which is a world proposition, since every proposition is $\top$ or $\perp$. And there is exactly one valuation, $\lambda pq. q = p$: any valuation V relates $\top$, the greatest lower bound of the empty property, to $\top$, and hence $\perp$ to $\perp$, and is therefore coextensive with $\lambda pq. q = p$, and identical to it by Extensionality. $\square$

This proves [Theorem D](#). In this model the logical atomist's picture holds exactly at the worlds. There, every assignment of truth values to the infinitely many elementary propositions determines a unique world, and conversely. Yet logical space also contains the atomless cloud, a region where nothing is elementary and every question is unsettled until refined. The elementary propositions are not rigid: they are elementary at every world and at no cloud point.

Remark 7.5. The infinitude of Λ is used only to show that the complete descriptions in terms of the elementary propositions contain no cloud points. With Λ finite, W_{V_0} would contain a non-empty part of the cloud, and AS would fail. Because every world sees every other, the worlds are modally on a par: each is the world according to exactly one truth-value assignment, and the actual world is distinguished only by being the evaluation point.

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