Coalesced Sums and Completeness for HME

Andrew Bacon*

November 26, 2018

What follows is a minimally edited draft of a paper I wrote and abandoned in 2018, originally titled just 'Coalesced Sums'. It outlines a semantics for Classicism and proves soundness and completeness, maximalization and disjunction properties of Classicism.[1] The models and axiomatic system are originally outlined in *The Broadest Necessity* (Bacon 2018), but there I only defined 'full' versions of the models. Similar results were later obtained in using action models described in *Classicism* (Bacon and Dorr 2024). However, the kind of model, and the proof method used here are sufficiently different to the published results that I thought it would be worth putting this online.

Notes on terminology. HME was my original name for a system equivalent to what is now known as Classicism. It refers to the axiom system H with the Rule of Equivalence and the principle Modalized Functionality, introduced in *The Broadest Necessity*. It is different from that system in one key respect that is not reflected by the name: it restricts the type system to relational types (types not ending in an e), which is also a restriction adopted later in *Classicism*. Here I call a logic "stable" for what in *Classicism* is called having a consistent maximalization.

***

In this paper I investigate theories of propositional granularity formulated in higher-order logic. By contrast to broadly structural theories of propositions, for example, the theories of propositional granularity explored here individuate propositions comparatively coarsely. All the theories I explore entail that propositions satisfy the Boolean identities, for instance. However, these theories are unlike many familiar Boolean theories of propositions, such as the extensional or possible worlds theory of propositions. They are rather inspired by a model in which propositions are individuated by *logical equivalence* in some designated logic and signature representing the fundamental properties and relations. Underlying the picture is the thought that, not only are logically equivalent propositions the same — something satisfied even in extensional models, for any reasonable notion of logical equivalence — but that logically inequivalent propositions are different.

*Thanks to the audience at the 2018 Grain exchange workshop for feedback on this. Some of the introductory material on higher-order logic in section 2 is reproduced from [1].

[1] These are the results reported in the appendix of *Logical Comibnatorialism*, Bacon 2020.

1 Introduction

This section motivates the results found in later sections. I will assume familiarity with the basic notation of higher-order logic throughout this section; the precise details of the language will be presented in section 2.

The theories investigated here are all extensions of a particularly natural higher-order theory: HME.[2] Very roughly, Booleanism is the thesis that propositions have the structure of a Boolean algebra and that disjunction, conjunction and negation express, respectively, the finitary join, meet and complement operations on this algebra. The thesis is stated at a level of generality where it implies that properties, operators, and other entities belonging to other relational types also have Boolean structure. HME extends Booleanism with the thesis that propositions are closed under a generalised (modal) notion of meet and join, indexed by other types, and that the logical quantifiers express these meets and joins.[3] HME is a natural extension of Booleanism: the former stands to the logical quantifiers as the latter stands to the logical connectives.

HME is a component of many familiar approaches to higher-order logic. For example, it is a part of the Fregean approach to higher-order logic in which there are only two propositions, *the true* and *the false*, and the possible worlds semantics, outlined e.g. in Gallin [3]. But, somewhat surprisingly, HME is consistent with unfamiliar visions of propositional granularity that are quite different from these two well-known instances of it.

Let me begin by extolling some of the virtues of HME that single it out as being particularly worthy of attention. The logical constants of a higher-order language are comprised of the truth functional operations and quantifiers enabling one to quantify into each grammatical position. Although these logical constants are all extensional, this does not preclude the existence of intensional operators, or even the possibility of defining intensional operators out of the extensional primitives. Theories extending HME come with their own internal modality, which can be defined out of the logical vocabulary: a proposition is necessary iff it is the same as the Boolean tautology, $\top$, where 'being the same proposition as' is cashed out quantificationally in terms of sharing the same higher-order properties.[4] The modality in question can be defined $\lambda p \forall X(X\top \rightarrow Xp)$. While there is much debate as to what exactly metaphysical necessity is, and over whether it can be reduced to other non-modal notions, the the meaning of the above modality is completely fixed by the meanings of the higher-order quantifiers and connectives and is clearly reducible to them since it is defined out of them. Moreover, the above modality occupies a particularly special logical role: it is the broadest modality, in the sense that any other operator that has a claim to being a modality of some sort is entailed by that operator.[5] We thus refer to it as *broad necessity* in what follows, and we write it $\Box$. HME entails that it is subject to the laws of a normal modal operator, and that it moreover has a logic of at least S4. Together these facts single out broad necessity as a powerful instrument for metaphysical theorizing.

Another favorable aspect of HME is that the logical connectives and quantifiers are uniquely pinned down by their logical roles. To illustrate, consider an alternative structural vision of propositions. The informal picture posits many operators. Do we have any way of specifying which of these operators is negation, for example? There are certainly many

[2] Defined in [1]. M here stands for Modalized Functionality.

[3] This is just an impressionistic gloss: there are some important differences, to be explained later, between the adjunctive notion of a meet or join in contrast to the usual order-theoretic notion.

[4] See [1] for further discussion.

[5] See [1, §4].

operators that do not fit the bill: necessarily, the negation of a proposition is true iff the negated proposition is not, and this constraint is violated by many operators.[6] But the constraint that negation reverses truth value certainly doesn't narrow the operations down to a single operation either. For if an operator X satisfies that logical role, then so does the operator that results from performing X thrice ($\lambda p\, XXXp$), or any odd number of times for that matter. According to the structural vision these are different operations — negation and triple negation, for instance, are different operators in virtue of being structurally different, even though they are necessarily equivalent. One might hope to rule out this proliferation by additionally requiring that negation be logically simple. But even this may not be enough: structural theories of propositions motivated by Frege's puzzle might wish to postulate many necessarily equivalent operators, all necessarily having the logical role of negation, that play different cognitive roles. The situation in HME, by contrast, is very different. Suppose, X is any operator and suppose it's broadly necessary that X applies to a proposition p iff that proposition is not the case. Then one can show that X must be negation. That is we can prove the following theorem:

$\neg$-role $\forall X(\Box\forall p(Xp \leftrightarrow \neg p) \rightarrow X = \neg)$

Analogous theorems may be derived for the other logical connectives and the logical quantifiers (see section 2.4). For example, if a connective necessarily applies to a pair of propositions iff they are both true then that connective is conjunction, and if a quantifier necessarily applies to a property Y iff Y applies to everything, then that quantifier is the universal quantifier:

$\wedge$-role $\forall X(\Box\forall pq(Xpq \leftrightarrow p \wedge q) \rightarrow X = \wedge)$

$\forall_\sigma$-role $\forall X(\Box\forall Y(XY \leftrightarrow \forall_\sigma Y) \rightarrow X = \forall_\sigma)$

It is also worth emphasizing that that HME is more parsimonious than the structural alternative. There are infinitely many truth functional connectives, but according to HME every truth-functional connective can be defined from a finite basis of connectives, say, $\neg$ and $\vee$. For example, it's common to define conjunction from negation and disjunction as follows $\wedge := \lambda p\lambda q\neg(\neg p \vee \neg q)$. Even if one introduced a primitive conjunction connective to the language, subject to the Boolean laws, one would be able to prove this identification using the principle $\wedge$-role, and so its addition would not allow one to express connectives that couldn't be expressed without it. This is not so for the structured theorist. If we impose the simplicity requirement, introduced above, then the structural theorist cannot make this identification, since a logically complex connective, like our proposed definition of conjunction, cannot be identical to a logically simple connective, like conjunction. One needs a separate logically simple connective to express conjunction, which while necessarily equivalent to $\lambda p\lambda q\neg(\neg p \vee \neg q)$ is distinct from it. These remarks apply equally to any truth functional connective. The structured theory is therefore under considerable pressure to posit infinitely many distinct logically simple connectives, one for each truth functional operation, such that no one of which is definable from any other.[7]

[6] The structural theory of propositions does not admit a well-behaved notion of broad necessity of the sort described above, but such theorists typically help themselves to a primitive notion of metaphysical necessity.

[7] Other options are available: one could simply maintain that some truth-functional roles are *special*, because they are occupied by logically simple entities, while others are not. But then there is a charge of arbitrariness to attend to: for example, if I insisted that the roles of negation and disjunction were occupied

Although HME is part of the familiar Fregean (extensional) and the intensional possible worlds accounts of propositions, it is also consistent with some less familiar metaphysical visions. Here I am particularly interested in theories that, informally speaking, individuate propositions by equivalence in some logic, relative to some choice constants representing the fundamental properties and relations. Unfortunately 'logical equivalence' is not something we can readily apply to propositions; it is a relation between sentences, and so this gloss can only take us so far. We might be a little more explicit about the picture by considering the following toy model. (The following may be seen to fix the intended equational theory — telling us which sentences of the form $\alpha = \beta$ are true or false — but as we shall note shortly, falls short of a full model of higher-order logic.) Fix a higher-order language in some signature, Σ, of non-logical 'fundamental' constants and some theory in this higher-order language, T (as already noted, we will be concentrating on theories that extend HME). We might then model propositions as equivalence classes of sentences of this language where sentences A and B belong to the same class when T proves $A = B$. We may similarly treat entities of other types in the same way, as equivalence classes of closed expressions of that type that are provably identical in T.

The choice of signature plays an important role in this construction, and the elements of the model corresponding to equivalence classes of constants enjoy distinctive logical properties. Firstly, note that the model construction is evidently signature dependent. Which propositions there are — i.e. which equivalence classes of sentences there are — depends on what fundamental constants are in the language. Secondly, it's a picture in which every proposition can be constructed out of fundamental entities (equivalence classes of non-logical constants) and logical operations (the equivalence classes closed terms only involving logical constants). It is in interesting picture because it seems to represent a metaphysical vision in which there are is a special class of properties, relations, etc. — the equivalence classes of fundamental constants — that freely generate the other properties, relations and so on. Finally, in the context of HME, the fundamental entities have a distinctive logical property: they are freely recombinable, in the sense that if a sentence ϕ containing fundamental constants $c_1...c_n$ is consistent with the theory T, then the corresponding proposition (the equivalence class of ϕ) is distinct from the inconsistent proposition (the equivalence class of $\bot$). This fact is suggestive, and reminiscent of a version of the Humean thesis that for any logically consistent way for the fundamental properties and relations to be, it's possible that those properties and relations be that way. This model-theoretic feature corresponds to the Humean thesis provided that possibility is understood as broad possibility — i.e. the modality of being distinct from the inconsistent proposition $\lambda p(p \neq \bot)$.

The above aspects should be contrasted with the extensional and possible worlds model theory in which no such collection of entities is distinguished in this way.

Structures like the one described above are a called *term models*, and are a standard tool in completeness arguments.[8] However, the picture of propositional granularity exhibited by such models when reasoning about them from a metatheoretical perspective is not always reflected in the object language. This is, in fact, a general phenomenon arising in models of higher-order logic: one can have models containing distinct elements a and b where the model 'thinks' of the elements as identical (i.e. makes the formula $x = y$ true relative to an

by simple connectives, but not conjunction, it's natural to ask why it is this way around and not some other. A less arbitrary option is to take all the unary and binary truth functional operations as basic, but not the ternary or higher arity operations (or, indeed, to take only the n-ary operations as primitive for any $n \geq 2$; note that $n \geq 2$ because the unary and 0-ary operations are not truth functionally complete). But this hardly avoids the arbitrariness charge altogether.

[8] [4]

assignment mapping x to a and y to b). Thus, even if we have built a model of higher-order logic around a term model, there is no guarantee that $\alpha \neq \beta$ will be true in the model when α and β are inequivalent in T and correspond to distinct equivalence classes. Term models help us get a fix on the intended equational theory, but do not settle the issue of the consistency of such theories within the background of higher-order logic. That is, while they fix the truth values of equations these models do not settle the truth values of arbitrary sentences, and so do not answer the question of whether it's possible to settle these truth values in a consistent manner. In fact, there a theories for which there are no models of higher-order logic that reflect the equations validated by the theories term model.

Let us now consider some object language principles that capture the sort of metaphysical vision described above. A natural first parse is a schema that simply asserts the distinctness of pairs of closed sentences that are not provably identical in T:

$Grain(T)$ $\alpha \neq \beta$ where α and β are closed terms of the same type not provably identical in T

More explicitly, the schema is just the set of sentences $\{\alpha \neq \beta \mid T \nvdash \alpha = \beta\}$. Note that schemas like $Grain(T)$ do not in general constitute recursive axiomatizations, as it may be an undecidable matter which equations T proves even when T is recursively axiomatizable.

When the above schema is consistent in T, we shall say that T is *stable*. Roughly this means that T is consistent with the idea that propositions, properties, etc, are individuated by logical equivalence according to the logic T. As we'll illustrate below, not all logics are stable.

A logic has the $=$-disjunction property, $DP(=)$, iff it admits the following rule:

Disjunction Property(=) If $\vdash_T A_1 = B_1 \vee ... \vee A_n = B_n$ then $\vdash_T A_i = B_i$ for some $i \in \{1...n\}$.

Here A_i and B_i are closed terms. The connection between stability and the disjunction property is made evident by the following proposition

Proposition 1. *1. Every stable logic has the disjunction property*

2. Every consistent compact logic with the disjunction property is stable.

For the first claim, suppose that T is stable and that $\vdash_T A_1 = B_1 \vee ... \vee A_n = B_n$. Suppose for contradiction that $\nvdash_T A_i = B_i$ for each $i \in \{1...n\}$. Then the set $A_1 \neq B_1, ..., A_n \neq B_n$ is consistent in T (since it is a subset of the consistent schema $Grain(T)$), contradicting our assumption that $\vdash_T A_1 = B_1 \vee ... \vee A_n = B_n$.

For the second claim, suppose $A_1 \neq B_1, ..., A_n \neq B_n$ is a finite set of distinctness claims from $\{A \neq B : \nvdash_T A = B\}$. Since $\nvdash_T A_i = B_i$ for any $i \in \{1...n\}$ we know that $\nvdash_T A_1 = B_1 \vee ... \vee A_n = B_n$ by the disjunction property. Thus $A_1 \neq B_1, ..., A_n \neq B_n \nvdash_T \bot$ and that set is consistent. By compactness the whole set $\{A \neq B : \nvdash_T A = B\}$ is consistent.

In HME a theory has $DP(=)$ iff it has the modal disjunction property, $DP(\Box)$:

Disjunction Property($\Box$) If $\vdash_T \Box A_1 \vee ... \vee \Box A_n$ then $\vdash_T \Box A_i$ for some $i \in \{1...n\}$.

As before, each A_i is closed. Given our definition of $\Box$, a sentence of the form $\Box A$ is equivalent to $A = \top$, and so it is evident that $DP(=)$ implies $DP(\Box)$. The converse is also derivable in HME because every identity statement is equivalent to its own necessitation,

and so a disjunction of identity claims is equivalent to a disjunction of necessitated claims, allowing you to apply $DP(\Box)$.[9]

It's worth emphasizing that not every logic is stable or has the disjunction properties.

Example 1. *Consider a higher-order language over a signature that contains two distinct constants, a and b, of type σ. Both the Fregean and the standard possible worlds model theory validate* HME*, and moreover validate:*

$$((a =_\sigma b) =_t \top) \vee (\neg(a =_\sigma b) =_t \top))$$

But neither semantics validate $(a =_\sigma b) =_t \top$ or $\neg(a =_\sigma b) =_t \top$): one can make $a = b$ true or false, as desired, in the latter case by constructing a model with two or more elements in type σ (such that the model also 'thinks' they are distinct) and interpreting a and b with these different elements.

The above example illustrates a failure of both $DP(=)$ and $DP(\Box)$ simultaneously, since the disjunction is, modulo definitions, $\Box(a =_\sigma b) \vee \Box\neg(a =_\sigma b)$. In a similar spirit, both model theories validate the principles of S5 *for $\Box$ and so validate:*

$$\Box\Diamond P \vee \Box\neg\Diamond P$$

where P is a fundamental constant of type t. But neither $\Box\Diamond P$ nor $\Box\neg\Diamond P$ are valid on either.

These two features demonstrate that when the signature of fundamental constants is sufficiently rich, stable logics will have to have two unfamiliar features that distinguish them from the extensional Fregean and the standard intensional approaches. Firstly, they will have to countenance contingent distinctness, for the following schema will consist of instances of $Grain(T)$ for any consistent substitution-closed extension T of HME

$$\Diamond a =_\sigma b$$

Assume that T is closed under the rule of substitution. Then $T \nvdash (a =_\sigma b) =_t \bot$, for if it did, then we could prove by the rule of substitution that $(a =_\sigma a) =_t \bot$, and then $\bot$ from $a =_\sigma a$ and Leibniz's law. So $\neg((a =_\sigma b) =_t \bot)$ is an instance of $Grain(T)$, which modulo definitions, is $\Diamond a =_\sigma b$. Suppose, moreover, that T does not prove that type σ is trivial: $T \nvdash x =_\sigma y$. Since T is closed under the rule of substitution, it follows that $T \nvdash a =_\sigma b$, and thus $T \nvdash (a =_\sigma b) = \top$. By similar reasoning we may reason that $\Diamond\neg(a =_\sigma b)$ is also an instance of $Grain(T)$.

For similar reasons, stable logics will not in general prove the B axiom, or various weakenings like B^n and B^* below:[10]

$$\mathsf{B} \quad A \to \Box\Diamond A$$

$$\mathsf{B}^n \quad A \to \Box\Diamond^n A$$

$$\mathsf{B}^* \quad \Box(A \to \Box A) \to \Diamond A \to A$$

each of which is sufficient for proving the necessity of distinctness. For example, B would entail the validity of $\Box\Diamond P \vee \Box\neg\Diamond P$.

Finally, it's worth noting that when the signature is trivial we can fairly easily construct stable logics that have neither of these properties:

[9] Indeed, this is derivable in any extension of higher-order theory that proves $(x = x) = \top$.

[10] Here $\Diamond^n$ represents a sequence of n $\Diamond$s.

Example 2. *Let M be a single model of higher-order logic in the purely logical signature (i.e. no non-logical constants). Let L be the theory of M in that signature: the set of sentences that are true in M no matter how we interpret the non-logical constants (of which there are non). Then L is stable. Because there are no non-logical constants, if $\phi = \psi$ is true in M then this identity is in the theory of M. So the negation of any identity not in the theory of M is true in M, and thus M validates $Grain(T)$.*

M in the above example could thus be a fixed standard extensional model, in which both B and the necessity of distinctness are validated. The theory of a single model in the empty signature is a peculiar edge case, however, and changes to the signature or number of models can make the logic unstable. For example, if we took the theory of all standard extensional models then, for example 'there are at least five individuals' is either $\top$ or $\bot$ in any given model, even though neither identity is part of this theory. Similarly, suppose we take a non-empty signature with a propositional constant, then neither $P = \top$ not $P = \bot$ is part of the theory of even a single Fregean model (because we can reinterpret P to be either).

2 Higher-order Logic

2.1 Simple Type Theory

In this section I present a version of Church's simple theory of types. Type theory allows to formulate theories that employ a variety non-logical devices, including familiar things from predicate logic, such as first-order predicates and singular terms, and but also non-logical connectives, predicate modifiers, quantifiers, and many others. These devices are subject to syntactic rules: an operator can combine with sentences but not predicates or names, and forth. Type theory provides a general framework which allows us to systematize these relationships. There are two *base* types, e and t, corresponding to the types of singular terms and sentences respectively. We restrict the type system to relational types: e and t are types, and $(\sigma \to \tau)$ is a type whenever σ and τ are types and $\tau \neq e$. Every type other than e therefore has the form $\sigma_1 \to \cdots \to \sigma_n \to t$, allowing $n = 0$. A functional type $(\sigma \to \tau)$ corresponds to expressions that take an argument of type σ and have a result of type τ.

We may illustrate this with a few examples. An operator combines with a sentence to produce a sentence, so will be assigned the type $(t \to t)$ in this system. We can also form higher-order predicates: expressions that combine with expressions of a complex type. For example, a quantifier expression, such as *some people*, combines with a predicate to produce a sentence (as in *some people run*), and so will have the type $((e \to t) \to t)$ which takes expressions with the complex type $(e \to t)$ as arguments. All expressions belonging to functional types have a single argument. Relations and connectives that take more than one argument are modeled by 'Currying': a binary relation is represented by the type $e \to (e \to t)$ and a binary connective by type $t \to (t \to t)$. An expression of the former type takes two singular terms in succession to make a sentence, and the latter two sentences.

We will follow the standard convention of omitting outer brackets and right-associated brackets from types (so, e.g., $t \to t \to t$ is an abbreviation for $(t \to (t \to t))$, but $((t \to t) \to t)$ only abbreviates to $(t \to t) \to t$).

A typed language is constructed from a signature of basic constants at each type: a type-indexed family of sets Σ^σ, where each Σ^σ consists of constants of type σ. We may now

formally define the terms over a signature Σ and their associated types.

1. M is a term of type σ if $M \in \Sigma^\sigma \cup Var^\sigma$
2. (MN) is a term of type τ if M has type $\sigma \to \tau$ and N type σ
3. $\lambda x\, M$ is a term of type $\sigma \to \tau$ if $x \in Var^\sigma$ and M is a term of type $\tau \neq e$.

If x has type σ and M type $\tau \neq e$, $\lambda x\, M$ is an expression that takes an expression of type σ as argument, N say, and results in an expression of type τ, with the meaning $M[N/x]$ — the result of substituting N for x in M. For example, suppose F and G are predicates (type $e \to t$), and consider $\lambda x(Fx \vee Gx)$ — the property of being F or G. The result of applying this constructed predicate to an argument a yields a sentence $(\lambda x(Fx \vee Gx))a$, which has the same meaning as $Fa \vee Ga$, as one would expect. In general, we should expect $(\lambda x\, M)N$ and $M[N/x]$ to have the same meaning, with the usual proviso that N doesn't contain any free variables that would get bound if substituted for x in M.[11] Any pair of expressions that can be obtained from the other by any number of substitutions of these two types of expressions, $(\lambda x\, M)N$ and $M[N/x]$, along with any relettering of bound variables, are called β-equivalent.[12] If M is an expression that doesn't contain x free, then $\lambda x(Mx)$ and M yield the same result when applied to any argument N: MN. Thus it's natural to think that these two expressions have the same meaning. Say that two expressions that can be obtained from one another by any number of substitutions of $\lambda x(Mx)$ for M or conversely, or by relettering of bound variables, are called η-equivalent. If they can be obtained from each other by using some number of either sort of operation, then they are $\beta\eta$-equivalent.

2.2 Higher-order Logic

Higher-order logic is formulated in a special signature that contains logical constants, including Boolean connectives, like $\wedge$ of type $t \to t \to t$, $\neg$ of type $t \to t$, and so on, and quantifiers $\forall_\sigma$ of type $(\sigma \to t) \to t$ for each type σ.[13] If F is a (possibly higher-order) predicate of type $\sigma \to t$, $\forall_\sigma F$ is a sentence that informally says that all things of type σ are F.[14] Thus higher-order logic allows quantification into any grammatical position. English only allows quantification into a limited set of grammatical positions, e.g. singular as in 'someone', plural as in 'some people' and prepositional as in 'somewhere', but not others like sentence position or operator position. So we must make do with approximations, for instance by pretending there is a class of 'things of type σ' and quantifying singularly over them. These paraphrases mustn't be taken too literally, and can lead one quickly into paradoxes if they are. However these paraphrases always single out a sentence of higher-order

[11] If this were to happen, one may reletter the bound variables in M to avoid this.

[12] If you can't substitute N for x in M without free variables getting bound then $(\lambda x\, M)N$ and $M[N/x]$ are not β-equivalent, but there is a relettering of M for which the corresponding pair is β-equivalent. This is why the relettering clause is included.

[13] One can express all truth functions using $\to$, $\forall_t$ and $\forall_{t\to t}$ since $\forall_{t\to t}\forall_t$ defines the same (0-ary) truth function as $\bot$, and the remaining truth functions may be expressed by combinations of $\to$ and $\bot$. However, it is more contentious whether you may literally define the other Boolean operations, like $\vee$, from these three constants (as opposed to merely express the truth functions expressed by the remaining Boolean operations). If connectives are individuated in a broadly structural way one might, for example, distinguish $\vee$ from $\lambda p\lambda q(\neg p \to q)$ on account of differences in structure. The structured theorist may then want to assume infinitely many primitive logical constants, one for each of the infinite number of truth functions.

[14] Here and throughout I am adopting the convention of calling things of type $\sigma \to t$ predicates. An operator of type $t \to t$ is thus a predicate in this more general sense, because it can be used to predicate something (e.g. necessity) of a proposition.

logic that is intended.[15] For a fuller defence of the intelligibility of higher-order logic itself I refer the reader to [5] and [6]. In addition to expressions of type σ, I shall thus also speak of entities of type σ, by which I mean entities that could be expressed by an expression of type σ.

We follow the usual convention of suppressing λs occurring after a quantifier, so that $\forall_\sigma \lambda x A$ may be abbreviated to the more familiar $\forall_\sigma x A$. We shall also write $=_\sigma$ for Leibniz identity — which we'll just pronounce *identity* — and is defined by $\lambda y \lambda z \forall_{\sigma\to t} X(Xy \to Xz)$. y and z are identical if they have the same properties.[16] We shall also frequently write schemata like $\forall_\sigma x\, x =_\sigma x$ which do not apparently involve any schematic letters. These are nonetheless schematic in their types: one should accept every well-formed instance of it when σ is replaced by a type. In general, when no ambiguity is possible, we will suppress type subscripts intending such schemata to have every well-formed insertion of type subscripts as instances.

We will be solely concerned with *higher-order* languages in what follows. A higher-order language contains a logical constant $\forall_\sigma$ of type $(\sigma \to t) \to t$ for each σ and a logical constant $\to$ of type $t \to t \to t$. Since all the signatures we will consider contain the logical constants, we write $\mathcal{L}(\Sigma)$ for the set of all terms over a signature Σ of *non-logical* constants and $\mathcal{L}^\sigma(\Sigma)$ for the terms of type σ. $\mathcal{L}(\emptyset)$ is thus the purely logical language, whose only constants are the logical constants.

To improve readability we introduce a number of abbreviations. $\bot$ stands for $\forall_{t\to t}\forall_t$, $\top$ for $\to \bot\bot$, $\neg$ for $\lambda p\,(\to p)\bot$, $\wedge$ stands for $\lambda p \lambda q \neg(\to p(\neg q))$, $\vee$ for $\lambda p\,(\to (\neg p))$, $\leftrightarrow$ for $\lambda p q(\wedge(\to pq)(\to qp))$, $=_\sigma$ for $\lambda x \lambda y \forall_{\sigma\to t} X(\to (Xx)(Xy))$ and $\Box$ for $(=_t \top)$ and $\Diamond$ for $\lambda p \neg\Box\neg p$. We also follow the convention of writing binary defined operations in infix position; e.g. writing $p \to q$ instead of $(\to p)q$, and $\top =_t p$ instead of $(=_t \top)p$.

A theory over a language $\mathcal{L}(\Sigma)$ is a subset of $\mathcal{L}^t(\Sigma)$. A standard way to present a theory is via a collection of axioms and rules. Here is a minimal theory of higher-order logic, which we call H:

Higher-order Logic

PC All instances of propositional tautologies.

MP From A and $A \to B$ infer B

Gen From $A \to B$ infer $A \to \forall_\sigma x B$ when x does not occur free in A.

UI $\forall_\sigma x \phi \to \phi[t/x]$ (where t is a term of type σ and no variable in t gets bound when substituted into ϕ)

$\beta\eta$ $A \leftrightarrow B$ whenever A and B are $\beta\eta$ equivalent terms of type t.

A propositional tautology is simply any term of type t with the form of a tautology. H is therefore the smallest set that contains every instance of a tautology, UI, and $\beta\eta$, and is closed under the MP and Gen rules.

[15] What is actually going on here is quite delicate. The paraphrasing sentences of English might well be strictly speaking false or nonsense, but their logical form allows us to identify a meaningful sentence of higher-order logic that is represented. One could alternatively talk in a quasi-technical language by modifying English grammar to accommodate certain sorts of quantification (see for example Prior's made-up English quantifier *anywhether* ([5])). Small modifications like this are common in mathematics and philosophy, especially when using made up technical vocabulary, but the modifications one would need to accommodate quantification into new grammatical positions would involve a significant divergence from intelligible English.

[16] Identity so defined can be shown to satisfy the usual axioms of identity already in the theory H described below.

2.3 HME

HME adds to H the principle Modalized Functionality, and closes the result under the Rule of Equivalence.

The Rule of Equivalence If $\vdash A \leftrightarrow B$ then $\vdash A =_t B$.

Modalized Functionality $\vdash \forall_{\sigma\to\tau} FG(\Box\forall_\sigma x(Fx =_\tau Gx) \to F =_{\sigma\to\tau} G)$.

Note that the rule of equivalence can be applied to proofs involving Modalized Functionality.

2.4 Broad Necessity

We shall use the following syntactic consequences of HME. Broad necessity is normal and satisfies S4; necessitation applies to theorems. Identity is persistent, propositional identity is necessary equivalence. That is both of the schemas below belong to HME

$$\begin{aligned} x =_\sigma y &\to \Box(x =_\sigma y),\\ A =_t B &\leftrightarrow \Box(A \leftrightarrow B),\end{aligned}$$

Also HME is closed under the rule of necessitation. If $A \in$ HME then $\Box A \in$ HME.

Only admitted types are used. These are standard results for HME; see [1, §§3–4, 7.5]. Repeated applications of Modalized Functionality, together with propositional identity as necessary equivalence, gives the logical-role principles in the introduction.

$\Gamma \vdash A$ means that $B_1 \wedge \cdots \wedge B_n \to A$ is a theorem for some finite list $B_1, \ldots, B_n \in \Gamma$. In particular, the Rule of Equivalence cannot be applied to arbitrary premises. All canonical theories below consist of closed sentences; when $\mathcal{L}^\sigma_w$ is used in that construction it denotes the closed terms of type σ.

3 Modalized Domains

Let $\mathcal{F}$ be a rooted transitive reflexive Kripke frame $(W, \leq, w_0)$. Informally, a modalized domain is a set D equipped with a collection world indexed relations, $\sim_w$, which intuitively tell you which things are identical at that world. For $a, b \in D$ we may think of a and b being identical at the world w if $a \sim_w b$. Existence is also contingent: $a \in D$ exists at w iff $a \sim_w a$ — if a is non-existent it does not stand in any identity relations, even to itself. As we have noted, the theories of interest permit distinct things to become identical later (with respect to the modal accessibility relation $\leq$), but once they are identical they remain identical. Similarly, the domain of existing entities may expand but not contract: once something exists it exists at all later worlds. Both facts are captured by the condition that: if $a \sim_w b$ and $w \leq w'$ then $a \sim_{w'} b$.

3.1 Modalized domains and homomorphisms

In what follows $\mathcal{F}$ always denotes a rooted transitive reflexive Kripke frame.

Definition 1. *A partial equivalence relation $\sim$ on a set D is a transitive symmetric relation on D.*

We define the domain *of $\sim$, $D(\sim)$, to be $\{a \in D \mid a \sim a\}$.*

$\sim$ clearly defines an equivalence relation on its domain. Thus we may equivalently think of a partial equivalence relation on a set D as an equivalence relation on a subset of D.

Definition 2 (Modalized Domain)**.** *A modalized domain over a frame $\mathcal{F} = (W, \leq, w_0)$ is an ordered pair $(D, \sim.)$ where*

- *D is a set.*
- *$\sim_w$ is a partial equivalence relation on D for each $w \in W$.*
- *If $a \sim_w b$ and $w \leq w'$ then $a \sim_{w'} b$.*

The inner domain *at w of D is the domain of $\sim_w$, $D(\sim_w)$, and we shall write it D_w.*

We follow the convention of referring to a modalized domain, $(D, \sim)$, by its first component D. We should think of, D_w, the inner domain at w as the set of things that exist at w, and $\sim_w$ as telling us which of the existing things are identical. The last condition ensures that we have expanding domains: if $w \leq w'$ then $D_w \subseteq D_{w'}$. It also ensures that we have the necessity of identity (but not necessarily of distinctness): if $w \leq w'$ and a and b are identical at w then they are identical at w'.

Definition 3. *A functional modalized domain is a modalized domain in which $\sim_w$ is an equivalence relation for each $w \in W$.*

It is also helpful to define the notion of a homomorphism between modalized domains.

Definition 4 (Homomorphism)**.** *If $(D, \sim)$ and $(D', \sim')$ are modalized domains over a frame $\mathcal{F} = (W, \leq, w_0)$, then a homomorphism is a function $f : D \to D'$ such that*

For every $w \in W$ and $a, b \in D$ if $a \sim_w b$ then $f(a) \sim'_w f(b)$

Two homomorphisms $f, g : D \to D'$ are essentially the same *iff for every world w and every $a, b \in D$, if $a \sim_w b$ then $fa \sim_w gb$.*

Modalized domains form a category whose objects are modalized domains and whose arrows are equivalence classes of homomorphisms that are essentially the same.

3.2 Function spaces and products

This category also has an exponential object, written $A \Rightarrow B$ which we will later use to interpret functional types.

Definition 5. *Given expanding modalized domains, $(A, \sim^A)$ and $(B, \sim^B)$, define $(A \Rightarrow B, \sim^{A \Rightarrow B})$ by:*

- *$A \Rightarrow B = B^A$, the set of all functions f from A to B.*
- *$f \sim_x^{A \Rightarrow B} g$ iff, for each $a, b \in |A|$ and each $y \in W$ such that $x \leq y$, if $a \sim_y^A b$ then $fa \sim_y^B gb$.*

To illustrate, suppose that A and B are modalized domains representing individuals and propositions respectively. The $A \Rightarrow B$ provide a natural model of properties of type $e \to t$. A property clearly cannot take a pair of objects that are 'identical' at w (bear $\sim_w$ to one another) to a pair of objects that are distinct at w, for that would represent a violation

of Leibniz's law. Thus the properties, f, that exist at a world w are those that preserve the identity relation from world w onwards. Moreover, in accordance with the principle of Modalized Functionality, f and g are counted as identical at w iff, at every accessible world w', they map identical things at w' to identical things at w'.

Other algebraic constructions on modalized domains are also possible. Here is another useful one:

Definition 6 (Product)**.** *The product of modalized domains $(A, \sim^A)$, $(B, \sim^B)$ is $(A \times B, \sim^{A\times B})$ where $\times$ is the usual cross product of two sets and:*

- $(a, b) \sim_w^{A\times B} (a\prime, b\prime)$ *iff* $a \sim_w^A a\prime$ *and* $b \sim_w^B b\prime$.

3.3 Subdomains

Definition 7. *Suppose that $(D, \sim^D)$ is a modalized domain. A modalized domain $(C, \sim^C)$ is a* subdomain *of D, written $C \leq D$, iff:*

$C \subseteq D$, and $C_w \subseteq D_w$ for every $w \in W$.

- *For every $w \in W$, and $a, b \in C_w$, $a \sim_w^C b$ if and only if $a \sim_w^D b$.*

Proposition 2. *$[f] : A \to B$ is monic if and only if, for any $w \in W$ and $a, a' \in A_w$, if $f(a) \sim_w f(a')$ then $a \sim_w a'$.*

Proof. Suppose that $a, a' \in A_w$ and $fa \sim_w fa'$. Let $C = \{0\}$ and $\sim_x = \{\langle 0, 0\rangle\}$ if $w \leq x$ and $\sim_x = \emptyset$ otherwise. Define $g, h : C \to A$ where $g(0) = a$ and $h(0) = a'$. g and h both preserve $\sim_x$ for every x: trivially if $w \not\leq x$, and if $w \leq x$ then $a \sim_x a$ and $a' \sim_x a'$ (since $a, a' \in A_w \subseteq A_x$). So g and h are arrows. Moreover $f \circ g$ is essentially the same as $f \circ h$: again trivially if $w \not\leq x$, and if $w \leq x$ then $f(g(0)) = f(a) \sim_x f(a') = f(h(0))$ (since $fa \sim_w fa'$ and $w \leq x$). Thus by monicity, $[g] = [h]$, i.e. if $0 \sim_x 0$ then $g(0) \sim_x h(0)$ for any $x \in W$. In particular, since $0 \sim_w 0$, $g(0) = a \sim_w a' = h(0)$ as required.

Conversely, suppose that $g, h : C \to A$ and $f \circ g$ and $f \circ h$ are essentially the same. Suppose also that for any $w \in W$, $a, a' \in A_w$, if $f(a) \sim_w f(a')$ then $a \sim_w a'$. We want to show that g and h are essentially the same. Suppose $c \sim_w c'$. So $f(gc) \sim_w f(hc')$ since $f \circ g$ and $f \circ h$ are essentially the same. Since $c \sim_w c$, and g is a homomorphism, $gc \sim_w gc$ and thus $gc \in A_w$. Similarly $hc' \in A_w$. Since $gc, hc' \in A_w$ and $f(gc) \sim_w f(hc')$ it follows, by our supposition, that $gc \sim_w hc'$ as required. So g and h are essentially the same, and f is monic.

□

3.4 Modalized Boolean algebras

In the following we outline a theory of Boolean algebras in the context of modalized domains.

Definition 8 (Modalized Boolean algebras)**.** *A modalized Boolean algebra over a frame $\mathcal{F}$ is a tuple $(B, \wedge, \neg)$ where*

- *B is a modalized domain*
- *$\wedge : B \times B \to B$ and $\neg : B \to B$ are homomorphisms*

  For every $w \in W$, if $a \sim_w a'$ then $\neg a \sim_w \neg a'$

For every $w \in W$, if $a \sim_w a'$ and $b \sim_w b'$ then $a \wedge b \sim_w a' \wedge b'$

- $(B, \wedge, \neg)$ *is a Boolean algebra.*

Note that the last condition could be weakened without affecting much to follow. Instead of requiring that each Boolean identity hold, we could merely insist that the Boolean equivalents $\sim$ each other at every world: e.g. in place of the identity $a \wedge b = b \wedge a$, we would only have the requirement that $a \wedge b \sim_w b \wedge a$ for every $w \in W$.

Given a frame $(W, \leq, w_0)$, $w \in W$ and $p \subseteq W$, we shall write $p \uparrow w$ to denote the set of worlds in p that are greater or equal to w: $\{w' \mid w' \in p \text{ and } w \leq w'\}$.

Example 3 (Modalized Boolean algebra a frame)**.** *Let $\mathcal{F} = (W, \leq, w_0)$ be a frame. The full modalized Boolean algebra over $\mathcal{F}$, written $B(\mathcal{F})$ is defined as follows.*

- $B(\mathcal{F}) := \mathcal{P}(W)$
- $p \sim_w q$ *if and only if* $p \uparrow w = q \uparrow w$
- $p \wedge q = p \cap q$
- $\neg p = W \setminus p$

In the above example we shall call elements $p \subset W$ *propositions*. If $\leq$ represents the accessibility relation for broad necessity, then $p \sim_w q$ represents the claim that p and q are broadly necessarily equivalent at w.

If B is a modalized Boolean algebra over $\mathcal{F}$, we shall write $a \leq_w b$ as short for $a \sim_w a \wedge b$. Suppose that I is an modalized domain, which we shall refer to as the 'index domain'.

Definition 9. *A modalized Boolean algebra B over a frame $\mathcal{F}$ is* complete *iff for any 'index' modalized domain I, there is a homomorphism $\inf_I : (I \Rightarrow B) \to B$ that satisfies the following adjunction property. For any $w \in W$, $b \in B_w$, $f \in (I \Rightarrow B)_w$*

- $b \leq_w \inf_I f$ *if and only if* $b \leq_x f(i)$ *for every $x \geq w$ and $i \in I_x$.*

It's worth noting that all infimum operations are essentially the same. For suppose that $\inf_I$ and $\inf'_I$ both satisfy the adjunction property. Suppose $f \sim_w g$. Since $\inf_I f \leq_w \inf_I f$ we know that for every $x \geq w$, $\inf_I f \leq_x f(i)$ for every $i \in I_x$, because inf satisfies the left-to-right direction of the adjunction property. Thus $\inf_I f \leq_w \inf'_I f$ since $\inf'_I$ satisfies the right-to-left direction. Similarly, we me reason that $\inf'_I f \leq_w \inf_I f$ and thus $\inf'_I f \sim_w \inf_I f$. Since $\inf'_I$ is a homomorphism, $f \sim_w g$ also gives $\inf'_I f \sim_w \inf'_I g$. Thus $\inf_I$ and $\inf'_I$ are essentially equivalent.

4 Models of HME

4.1 Frames and Models

Definition 10 (Frame)**.** *A higher-order frame over a Kripke frame $\mathcal{F}$ consists of a collection of modalized domains D^σ, indexed by the relational types, with $D^\sigma_w \neq \emptyset$ for every type σ and world w, such that:*

1. $D^t \leq B(\mathcal{F})$ *and* $\emptyset \in D^t_w$ *for every $w \in W$.*
2. $D^{\sigma \to \tau} \leq D^\sigma \Rightarrow D^\tau$ *for every admitted function type*

3. *The following combinators and logical operations belong to D^σ_v for every world v and σ of appropriate type:*

   (a) $k \in D^{\sigma\to\tau\to\sigma}_v$ *where* $k(a)(b) = a$ *for all* $a \in D^\sigma, b \in D^\tau$, *with* $\sigma \neq e$

   (b) $s \in D^{(\sigma\to\tau\to\rho)\to(\sigma\to\tau)\to(\sigma\to\rho)}_v$ *where* $s(a)(b)(c) = ac(bc)$ *for all* $a \in D^{\sigma\to\tau\to\rho}, b \in D^{\sigma\to\tau}, c \in D^\sigma$, *with* $\tau, \rho \neq e$

   In addition, for every admitted instance, the exchange and contraction combinators belong to the inner domain:

$$c : (\sigma \to \tau \to \rho) \to \tau \to \sigma \to \rho, \qquad c(f)(a)(b) = f(b)(a),$$
$$d : (\sigma \to \sigma \to \tau) \to \sigma \to \tau, \qquad d(f)(a) = f(a)(a).$$

   Here $\rho \neq e$ *in the first line and* $\tau \neq e$ *in the second.*

   (c) $if \in D^{t\to t\to t}_v$ *where* $if(p)(q) = (W \setminus p) \cup q$

   (d) $all_\sigma \in D^{(\sigma\to t)\to t}_v$ *where* $all_\sigma(f) = \{w \mid w \in fa$ *for every* $a \in D^\sigma_w\}$.

   (e) $eq_\sigma \in D^{\sigma\to\sigma\to t}_v$ *where* $eq^\sigma(a)(b) = \{w \mid a \sim^\sigma_w b\}$

Definition 11. *A frame is:*

- Functionally full *iff* $D^{\sigma\to\tau} = D^\sigma \Rightarrow D^\tau$ *for every admitted type* $\sigma \to \tau$
- Propositionally full *iff* $D^t = B(\mathcal{F})$.

Note that if D is functionally full and propositionally full clause 3, ensuring that the relevant logical operations are in the model, is no longer needed.

Definition 12. *Let* Σ *be a signature. A* Σ*-model over* $\mathcal{F}$ *is a pair* $(D^\cdot, [\![\cdot]\!]^\cdot)$ *where*

- $D^\cdot$ *is a frame over* $\mathcal{F}$
- $[\![c]\!] \in D^\sigma_w$ *when* $c \in \Sigma^\sigma$ *and* $w \in W$

A variable assignment is a function g with domain Var such that $g(x) \in D^\sigma$ when $x \in Var^\sigma$. If $x \in Var^\sigma$ and $a \in D^\sigma$, write $g[x \mapsto a]$ for the variable assignment exactly like g except it maps x to a. Given an assignment g, define $[\![M]\!]^g$ for arbitrary terms M by

- $[\![c]\!]^g = [\![c]\!]$
- $[\![\forall_\sigma]\!]^g = all_{D^\sigma}$
- $[\![\to]\!]^g = if$
- $[\![x]\!]^g = g(x)$
- $[\![MN]\!]^g = [\![M]\!]^g([\![N]\!]^g)$
- $[\![\lambda x M]\!]^g = a \mapsto [\![M]\!]^{g[x\mapsto a]}$

The combinators ensure closure under the admitted abstractions. In bracket abstraction, the usual k, s cases apply when the relevant intermediate result types are not e. If an application has an argument of type e, that argument is a variable or constant. An argument independent of the abstracted variable is handled by c, and a repeated occurrence of the abstracted individual variable by d. The identity abstraction is needed only at types other than e, where it is expressible using s, k. This gives the usual abstraction induction without introducing a function with result type e.

Recall that in a frame or Σ-model D^t consists of subsets of W. We may use this to define what it means for a term of type t to be true or false

Definition 13. *If $A \in \mathcal{L}^t(\Sigma)$ and $(D, [\![\cdot]\!])$ a Σ-model, then we say that A is satisfied by an assignment g admissible at w_0 iff $w_0 \in [\![A]\!]^g$. If A is closed we say it is true if it is satisfied by every assignment admissible at w_0.*

If g and h are assignments, say that $g \sim_w h$ iff $g(x) \sim_w h(x)$ for every $x \in Var$.

Proposition 3. *If $g \sim_w h$ then $[\![M]\!]^g \sim_w [\![M]\!]^h$.*

Proof. Use the combinatory translation just described. Variables and constants satisfy the claim, and application preserves it by the function-space condition. The translation has the same set-theoretic value as the original term. In particular, $[\![M]\!]^g \in D^\sigma_w$ whenever $g \sim_w g$. □

4.2 Broad necessity in frames

The defined $\bot$ denotes $\emptyset$. Indeed, the identity operator on t exists everywhere, and its universal closure is false at each world because $\emptyset$ belongs to that world's propositional domain. Hence $\forall_{t\to t}\forall_t$ is false at every world. It follows that $\top$ denotes W, and the defined Boolean connectives have their usual set-theoretic values.

For existing $a, b \in D^\sigma_w$, the defined Leibniz identity holds at w exactly when $a \sim_w b$. The forward implication uses the existing predicate $eq_\sigma(a)$: it applies to a and hence to b. For the converse, every existing predicate preserves $\sim_w$. Thus the stipulated eq_σ and the defined identity agree in truth value on existing arguments.

Proposition 4. *Suppose that D is a frame over $\mathcal{F}$. Then for any term A of type t, $x \in W$ and assignment $g \sim_x g$*

- *$x \in [\![\Box A]\!]^g$ if and only if for every $y \geq x$, $y \in [\![A]\!]^g$*

Proof. The defined $\Box A$ is $\top =_t A$. By the preceding observation it holds at x iff $W \sim^t_x [\![A]\!]^g$, which is precisely the displayed condition because $D^t \leq B(\mathcal{F})$. □

4.3 Soundness

Theorem 5. *Suppose that A is a Σ-model over the frame $(W, \leq, w_0)$, and g is an assignment mapping variables into $A^\sigma_{w_0}$ (i.e. $g \sim_{w_0} g$). Every theorem of HME is satisfied by g in A.*

Proof. We prove validity at every world under every assignment admissible there. Propositional tautologies and modus ponens are sound because implication has its usual truth conditions. Universal instantiation is sound because Proposition 3 places the value of the instantiating term in the current inner domain. Generalization is sound by varying the

assigned value of the quantified variable over that domain; the side condition ensures that the antecedent is unchanged. Set-theoretic interpretation validates $\beta\eta$-conversion, including conversion inside larger terms.

The Boolean equations hold because their fully applied values are identical subsets of W. Repeated use of the function-space equivalence and the preceding characterization of Leibniz identity therefore validates the equations between the abstracted operations.

For the first adjunction equation, let $f \in D_w^{\sigma\to t}$ and $a \in D_w^\sigma$. At any $v \geq w$, if $v \in all_\sigma(f)$ then $v \in f(a)$, since $a \in D_v^\sigma$. Hence $all_\sigma(f) \cup f(a) \sim_w f(a)$. For the second, if $p \in D_w^t$ then

$$p \cap all_\sigma(\lambda y.p) \sim_w p :$$

at every $v \geq w$ where p is true, every instance of the constant predicate is true. These arguments apply after any successive choices of accessible worlds for the displayed abstractions, so the function-space condition validates the two equations themselves. □

5 Completeness for HME

Here we establish the completeness of HME with respect to the models described above.

Let S be a consistent set of sentences over HME in a countable language $\mathcal{L}$. Let $\mathcal{L}_\infty$ be the result of adding countably infinitely many constants to $\mathcal{L}$ at each type. We will be interested in sets w such that:

> w is maximally consistent (with HME) and is witness complete relative to a language $\mathcal{L}_w$, which contains $\mathcal{L}$, is contained in L_∞, and is such that there are infinitely many constants in $\mathcal{L}_\infty$ that do not appear in $\mathcal{L}_w$ at each type. (We use $\mathcal{L}_w$ to always denote the minimal language that contains all the formulas in w.)
>
> $w \leq w'$ iff whenever $\Box A \in w$, $A \in w'$. (Note that this implies that $\mathcal{L}_w \subseteq \mathcal{L}_{w'}$).

Witness completeness means that for each closed $\phi : \sigma \to t$ in $\mathcal{L}_w$ there is a constant $c : \sigma$ in that language such that $\phi c \to \forall_\sigma \phi \in w$. Equivalently, failures of universal quantification have named counterexamples.

Every consistent set in an eligible language has such an extension in a larger eligible language. Split the unused constants at each type into two infinite sets, reserving one set permanently. Enumerate the sentences and universal formulas in the language obtained by adjoining the other set. At successive stages decide the next sentence consistently and add a counterexample-witness implication with a constant fresh for the original set and all sentences added so far. Adding that implication preserves consistency: otherwise a finite conjunction C of previous premises would imply both ϕc and $\neg\forall\phi$; replacing the fresh c by a variable and generalizing gives $C \to \forall\phi$, a contradiction. A fair enumeration also treats formulas containing the new witness constants. The union is complete and witnessed, and leaves the reserved constants unused.

The relation $\leq$ is reflexive and transitive by T and 4. It preserves languages because $\Box(c = c)$ belongs to w for each constant of its language, and hence $c = c$ belongs to every successor.

We define a higher-order frame as follows.

- $W = \{w \mid w \geq w_0\}$, where w_0, our root world, is some max con witness complete set containing S.

- $D^t = P(W)$
- $D^e = \mathcal{L}^e_\infty$

  $D^{\sigma\to\tau} = (D^\tau)^{D^\sigma}$.

As an intermediate step to defining the PER $\sim^\sigma_w$ at each world, we will define a world indexed class of 'tethering' relations between D and the language L_∞.

- $R^t_w pA$ iff $A \in \mathcal{L}_w$ and $p \uparrow w = \{w' \mid A \in w'\} \uparrow w$
- $R^e_w \alpha\alpha'$ iff $\alpha = \alpha' \in w$
- $R^{\sigma\to\tau}_w f\phi$ iff $\phi \in \mathcal{L}_w$ and for every $w' \geq w$, if $R^\sigma_{w'} a\alpha$ then $R^\tau_{w'}(fa)(\phi\alpha)$

We finally define our PER

- $a \sim^\sigma_w b$ iff for some α $R^\sigma_w a\alpha$ and $R^\sigma_w b\alpha$.

Proposition 6. *For $A \in \mathcal{L}^t_w$, $\Box A \in w$ iff $A \in w'$ for all $w' \geq w$.*

Proof. The forward direction is the definition of accessibility. If $\Box A \notin w$, the set $\{B : \Box B \in w\} \cup \{\neg A\}$ is consistent. Otherwise normality and theorem-level necessitation applied to a finite implication would put $\Box A$ in w. Extend this set to a complete witnessed theory using fresh constants as above. It is a successor of w, and hence lies above w_0, but does not contain A. □

Proposition 7. *For any type σ, world w and $\alpha, \alpha' \in \mathcal{L}^\sigma_w$:*

1. *There is some $a \in D^\sigma$ such that $R^\sigma_w a\alpha$*
2. *If $R^\sigma_w a\alpha$ then: $R^\sigma_w a\alpha'$ iff $\alpha = \alpha' \in w$*

Proof. First, $R_w a\alpha$ implies $R_v a\alpha$ whenever $v \geq w$. At e this uses persistence of identity; at t it is restriction of the truth set; at function types it follows by restricting the worlds quantified over.

Prove the two claims simultaneously by induction on types. At e, take $a = \alpha$; the equality claim follows from the laws of identity. At t, take $a = \{v \in W : \alpha \in v\}$. The equality claim follows from Proposition 6 and $\alpha = \alpha' \leftrightarrow \Box(\alpha \leftrightarrow \alpha')$.

For a non-base type write $\sigma = \sigma_1 \to \cdots \to \sigma_n \to t$. Define the total curried function f_α by

$$f_\alpha(a_1)\cdots(a_n) = \{v \geq w : \exists \beta_i \in \mathcal{L}^{\sigma_i}_v \ (1 \leq i \leq n) \text{ such that } R^{\sigma_i}_v a_i \beta_i \text{ for every } i, \quad \alpha\beta_1 \cdots \beta_n \in v\}.$$

If two tuples of terms represent the same arguments at v, the induction hypothesis identifies the corresponding terms in v. Substitutivity therefore makes the last truth condition independent of the chosen representatives. Persistence shows that the same verification works when successive arguments are supplied at successive accessible worlds. Consequently $R_w f_\alpha \alpha$. This construction uses the restriction to relational types: its final value is a subset of W.

For the equality claim at a function type $\rho \to \tau$, suppose $R_w f\alpha$ and $R_w f\alpha'$. At every $v \geq w$ and for every closed $\beta : \rho$ in $\mathcal{L}_v$, surjectivity at ρ supplies an argument representing

β. The equality claim at τ then gives $\alpha\beta = \alpha'\beta \in v$. Witness completeness yields $\forall x(\alpha x = \alpha' x) \in v$. Proposition 6 and Modalized Functionality give $\alpha = \alpha' \in w$. Conversely, such an identity persists to every $v \geq w$; substitutivity and the equality claim at τ then show that any representative of α also represents α'. □

Proposition 8. *D is a higher-order frame.*

Proof. Proposition 7 implies that the induced relations are symmetric and transitive, and the persistence of R implies persistence of these PERs. Each inner domain is nonempty, since the witnessed language contains a closed term of each type. At t, two existing propositions are equivalent exactly when they agree above the current world, and $\emptyset$ is tethered to $\bot$ at every world. We verify next the quantifier and function-space clauses.

To show the quantifier clause, we must show $all_\sigma \in D_w^{(\sigma\to t)\to t}$ for every w. It suffices to show that all_σ Rs something. We shall show that it Rs $\forall$. I.e. for every $w' \geq w$, if $R_{w'} f\phi$ then $R_{w'} all(f)\forall\phi$.

So suppose that $w' \geq w$, and $R_{w'} f\phi$. If $x \in all_\sigma(f) \uparrow w'$ then $x \in f(a)$ for every $a \in D_x^\sigma$. Let c be the witness of $\forall\phi$ in x. By the surjectivity of R, there's an $a \in D_x^\sigma$ such that $R_x^\sigma ac$. Since $R_{w'} f\phi$ and $x \geq w'$, $R_x f\phi$. So $R_x^t(fa)(\phi c)$, which means that $fa \uparrow x = \{y \geq x \mid \phi c \in y\}$. Since $x \in f(a')$ for any $a' \in D_x^\sigma$ it follows that $x \in fa \uparrow x$ and so $\phi c \in x$. Since c is the witness of $\forall\phi$ in x, $\phi c \to \forall\phi \in x$ and so $\forall\phi \in x$.

Conversely, suppose that $\forall\phi \in x$, and that $a \sim_x a$. The latter means that R_x is defined on a, so let $\alpha \in \mathcal{L}_x$ be such that $R_x a\alpha$. Since $\forall\phi \in x$, $\phi\alpha \in x$, and since $R_x f\phi$, $R_x(fa)(\phi\alpha)$ so $x \in fa$ as required.

To show the function-space clause suppose that $f, g \in D_w^{\sigma\to\tau}$, and that $fa \sim_{w'} gb$ whenever $w' \geq w$ and $a \sim_{w'} b$.

Since $f, g \in D_w^{\sigma\to\tau}$ it follows that there exists $\phi, \psi \in \mathcal{L}_w$ such that $R_w f\phi$ and $R_w g\psi$. In order to show $f \sim_w g$ it suffices to show that $\phi = \psi \in w$ (by Proposition 7). Since w contains Modalized Functionality, and since it is closed under MP, it suffices to show $\Box\forall x(\phi x = \psi x) \in w$. By Proposition 6 it suffices to show that $\forall x(\phi x = \psi x) \in w'$ for every $w' \geq w$. Finally, it suffices to show that for all $w' \geq w$ $\phi c = \psi c \in w'$ where c is the witness for w' of $\forall x(\phi x = \psi x)$.

By Proposition 7 we know that $R_{w'}^\sigma$ is surjective so there is an $a \in D_{w'}^\sigma$ such that $R_{w'}^\sigma ac$. We know that $a \sim_{w'} a$, and so by our initial assumption (that $fa \sim_{w'} gb$ whenever $w' \geq w$ and $a \sim_{w'} b$), it follows that $fa \sim_{w'} ga$. Since $R_w^{\sigma\to\tau} f\phi$ and $R_w^{\sigma\to\tau} g\psi$, $w' \geq w$ and $R_{w'}^\sigma ac$ we know that $R_{w'}^\tau fa\phi c$ and $R_{w'}^\tau ga\psi c$. So by Proposition 7 $\phi c = \psi c \in w'$ as required.

Conversely, suppose that $f \sim_w g$, $w' \geq w$ and $a \sim_{w'} b$. The first implies that for some ϕ, $R_w f\phi$ and $R_w g\phi$, and thus also $R_{w'} f\phi$ and $R_{w'} g\phi$, the second implies that for some α, $R_{w'} a\alpha$ and $R_{w'} b\alpha$, and so $R_{w'}(fa)(\phi\alpha)$ and $R_{w'}(gb)(\phi\alpha)$. So $fa \sim_{w'} gb$ as required.

The combinators are tethered to their defining closed lambda terms: unfold their applications and use the defining clause for R and β-conversion. This applies to c, d as well as the admitted k, s instances. Implication is tethered to $\to$, since membership in $(W \setminus p) \cup q$ agrees with the truth condition for implication in each complete theory. Finally, for $R_v a\alpha$ and $R_v b\beta$, Proposition 7 gives

$$a \sim_v b \iff \alpha = \beta \in v.$$

This equality at every accessible world tethers eq_σ to the defined identity. Thus all stipulated logical operations belong to every inner domain. □

Theorem 9 (Completeness)**.** *Every consistent set of sentences in a countable relational language has a model of the specified kind.*

Proof. For each constant c of the original language choose, by Proposition 7, a value $[\![c]\!]$ with $R_{w_0}[\![c]\!]c$. Persistence makes it admissible everywhere. More generally, induction on terms proves the following fundamental lemma: if each free variable x is assigned a value tethered at w to a closed term $\theta(x)$, then

$$R_w[\![M]\!]^g\, M[\theta].$$

The constant cases were just verified, and application follows directly from the definition of R. For abstraction, pass to any $v \geq w$, choose an argument tethered there to a closed term, and apply the induction hypothesis with that additional substitution. Persistence and capture-avoiding substitution justify this step.

For a closed sentence A of the original language the lemma gives

$$w \in [\![A]\!] \quad \Longleftrightarrow \quad A \in w.$$

In particular all sentences of $S \subseteq w_0$ are true at the root. Together with soundness this also establishes completeness for local consequence in countable languages. □

6 Coalesced Sums

Suppose that A^i, indexed by $i \in I$, are a collection of Henkin models, over frames $(W_i, \leq_i)$ where $W_i \cap W_j = \emptyset$ for $i \neq j$. Assume that I is a nonempty set. We write w_i for the root of W_i, and w_0 denotes a new world not in any of the frames.

- $W := \bigcup_i W_i \cup \{w_0\}$
- $\leq := \bigcup \leq_i \cup \{(w_0, w) \mid w \in \bigcup_i W_i \cup \{w_0\}\}$ (the coalesced sum of frames) .
- $D^e := \Pi_i A^{ie}$

  $D^t := P(W)$.

  $D^{\sigma\to\tau} := (D^\tau)^{D^\sigma}$.

  First we define a Kripke logical relation $R_w : D \to A^i$, where $w \in W_i$.

- For $w \in W_i$, $R^t_w pq$ iff $q \in A^{it}_w$ and $p \uparrow w = q \uparrow w$.

  For $w \in W_i$, $R^e_w (a_j)_{j\in I} b$ iff $a_i \sim^{A^{ie}}_w b$.

- $R^{\sigma\to\tau}_w fg$ iff $g \in A^{i\sigma\to\tau}_w$ and for every $w' \geq w$, if $R^\sigma_{w'} ab$ then $R^\tau_{w'}(fa)(gb)$.

For $w \neq w_0$ define

- $a \sim^\sigma_w b$ iff for some $c \in A^{i\sigma}_w$, $R^\sigma_w ac$ and $R^\sigma_w bc$.

At the new root define the relations recursively on types. At the base types put

$$\begin{aligned} p \sim^t_{w_0} q \quad &\Longleftrightarrow \quad p = q \ \text{ and } \ \forall i \in I\ \exists r \in A^{it}_{w_i}\ R^t_{w_i} pr, \\ a \sim^e_{w_0} b \quad &\Longleftrightarrow \quad a = b \ \text{ and } \ \forall i \in I\ a_i \in A^{ie}_{w_i}. \end{aligned}$$

At a function type put $f \sim^{\sigma\to\tau}_{w_0} g$ iff both of the following hold:

1. $f, g \in D^{\sigma\to\tau}_{w_i}$ for every $i \in I$;

2. for every $v \geq w_0$ and $a, b \in D^\sigma$, if $a \sim^\sigma_v b$ then $fa \sim^\tau_v gb$.

Here the branch inner domains are defined by the branch relations above. The second clause includes $v = w_0$. In particular, propositional equivalence at the new root requires agreement at that root as well as on the branches.

As before we prove:

Proposition 10. *For any type σ, world $w \neq w_0$ in W_i and $b, b' \in A^{i\sigma}_w$:*

1. *There is some $a \in D^\sigma$ such that $R^\sigma_w ab$.*

2. *If $R^\sigma_w ab$ then: $R^\sigma_w ab'$ iff $b \sim_w b'$.*

Proof. As in the canonical construction, R persists towards accessible worlds. Prove the two claims together by induction on types. At e, choose a tuple with ith coordinate b and arbitrary coordinates elsewhere; the second claim is the PER law in A^{ie}. At t, extend $b \subseteq W_i$ to the same subset of W; the second claim follows from $A^{it} \leq B(W_i)$.

At a relational type $\sigma_1 \to \cdots \to \sigma_n \to t$, represent $b \in A^{i\sigma}_w$ by the curried function

$$F_b(a_1)\cdots(a_n) = \{v \in W_i : v \geq w,\ \exists b_j \in A^{i\sigma_j}_v\ (1 \leq j \leq n) \text{ such that } R^{\sigma_j}_v a_j b_j \text{ for every } j, \quad v \in b(b_1)\cdots(b_n)\}.$$

The second induction hypothesis makes this truth condition independent of the representatives, since existing functions in A^i preserve equivalence. Persistence gives the tethering condition even when successive arguments are supplied at different accessible worlds.

For the second claim at $\rho \to \tau$, if one f represents both b and b', represent each argument $c \in A^{i\rho}_v$ by surjectivity at ρ. The equality claim at τ gives $bc \sim_v b'c$ at every $v \geq w$. Preservation of equivalence by b' also yields the corresponding assertion for equivalent arguments c, c', and the function-space condition in A^i gives $b \sim_w b'$. Conversely, if $b \sim_w b'$, use that function-space condition and the equality claim at τ in the definition of R. □

Proposition 11. *D is a higher-order frame.*

Proof. On each branch, Proposition 10 makes the induced relations persistent PERs and gives the restricted propositional equivalence condition. We first verify the function-space and quantifier clauses on those branches.

Suppose $f, g \in D^{\sigma\to\tau}_w$ and for all $w' \geq w$, if $a \sim_{w'} b$ then $fa \sim_{w'} gb$. Since $f, g \in D^{\sigma\to\tau}_w$, it follows that there exists $h, h' \in A^{i(\sigma\to\tau)}_w$ with $R_w fh$ and $R_w gh'$. To show that $f \sim_w g$ (in D) it suffices to show that $h \sim_w h'$ (in A^i). Suppose $w' \geq w$ and $c \sim_{w'} c'$. By surjectivity, there are $a, b \in D^\sigma_{w'}$ such that $R_{w'} ac$ and $R_{w'} bc'$. Putting this together we get that $R_{w'}(fa)(hc)$ and $R_{w'}(gb)(h'c')$.

Note also that since $c \sim_{w'} c'$, $a \sim_{w'} b$, and by our initial supposition $fa \sim_{w'} gb$. So there is an r such that $R_{w'}(fa)r$ and $R_{w'}(gb)r$, and lastly by Proposition 10, $hc \sim_{w'} r$ and $h'c' \sim_{w'} r$. So $hc \sim_{w'} h'c'$, and thus $h \sim_w h'$ as required.

For the converse, if f, g represent a common h and a, b represent a common argument c at $w' \geq w$, then fa, gb represent hc. Hence $fa \sim_{w'} gb$.

For the quantifier, we show $R_w all_\sigma\, all^i_\sigma$. If $w' \geq w$ and $R_{w'} fg$, then at every $v \geq w'$,

$$\big(v \in f(a) \text{ for every } a \in D^\sigma_v\big) \quad \Longleftrightarrow \quad \big(v \in g(b) \text{ for every } b \in A^{i\sigma}_v\big).$$

For the forward direction lift each b by surjectivity; for the reverse direction choose a representative b for each existing a. In either case $R_v(fa)(gb)$ equates their truth values at v. Thus the two quantified propositions agree throughout the cone above w'.

The combinators and implication also represent the corresponding component operations. Proposition 10 gives $a \sim_v b$ iff their representatives are equivalent in A^i, so eq_σ represents the component identity operation. This verifies all logical-operation requirements on the branches.

At the root, induction on types shows that the specified relations are PERs. For transitivity at a function type, use reflexivity on an existing argument and transitivity at the result type; symmetry is similar. Root equivalence implies branch equivalence: at function types this follows from branch admissibility and the branch function-space clause just proved. Thus the relations persist from the root. Propositional equivalence there is literal equality, as required because the root sees all of W.

For existing root functions the defining root clause is exactly the required function-space equivalence. Each combinator is admissible on every branch and preserves the root relations by its defining application equation. Implication preserves root equivalence because equivalent root propositions are identical subsets of W.

For the quantifier, suppose $f \sim_{w_0}^{\sigma\to t} g$. The branch argument already makes $all_\sigma(f)$ and $all_\sigma(g)$ equivalent on every branch. For each $a \in D_{w_0}^\sigma$ the root clause gives $f(a) \sim_{w_0}^t g(a)$, hence equality of these subsets. The two quantified propositions therefore also agree at w_0, and so are identical subsets of W. Together with branch admissibility this proves $all_\sigma \in D_{w_0}^{(\sigma\to t)\to t}$.

For identity, if $a \sim_{w_0} a'$ and $b \sim_{w_0} b'$, the PER laws and persistence give

$$a \sim_v b \quad\Longleftrightarrow\quad a' \sim_v b' \qquad (v \in W).$$

Hence $eq_\sigma(a)(b) = eq_\sigma(a')(b')$ as subsets of W. Branch admissibility and the root function-space clause show that eq_σ exists at the root as well.

Finally, the root individual domain is the nonempty product $\prod_i A_{w_i}^{ie}$. The propositions $\emptyset, W$ exist there, and the combinators give an existing constant predicate with value W at every type other than e. All inner domains are therefore nonempty, and $\emptyset$ exists at every world. □

Proposition 12 (Lifting component interpretations)**.** *For each family $b_i \in A_{w_i}^{i\sigma}$ there is an $a \in D_{w_0}^\sigma$ with $R_{w_i}^\sigma a b_i$ for every i.*

Proof. At e take the tuple $(b_i)_i$; at t take $\bigcup_i b_i$, with either choice of truth value at w_0. At a function type, use the formula in the proof of Proposition 10 simultaneously on all branches, with $w = w_i$ and $b = b_i$, taking the union of the resulting truth sets and making every fully applied output false at w_0. It is tethered to b_i on each branch. To check root admissibility, supply root-equivalent arguments. They are equivalent on every branch, so the final outputs agree on every branch; they also agree at w_0, where both are false. Working backwards through the curried arguments verifies the recursive root function-space clause, including branch admissibility of partial applications. □

Theorem 13 (Coalesced sums)**.** *Every nonempty set-indexed family of models in a common relational signature has a model on its coalesced frame which preserves the truth of every sentence at every world of each component.*

Proof. Use Proposition 12 to interpret each non-logical constant by a common lift of its component interpretations. The logical operations are already tethered to their component counterparts. A term induction, with related assignments, now gives

$$R^{\sigma}_{w}[\![M]\!]^{g}_{D}\,[\![M]\!]^{h}_{A^i} \qquad (w \in W_i).$$

For abstraction, pass to any future world and any related pair of arguments and apply the induction hypothesis with the assignments extended. At type t this relation implies agreement of truth values at w. For closed sentences no assignment parameters remain, giving the assertion. □

References

[1] Andrew Bacon. "The Broadest Necessity." *Journal of Philosophical Logic* 47 (2018), 733–783.

[2] Cian Dorr. "Quantifier Variance and the Collapse Theorems." *The Monist* 97 (2014), 503–570.

[3] Daniel Gallin. *Intensional and Higher-Order Modal Logic*. North-Holland, 1975.

[4] Leon Henkin. "Completeness in the Theory of Types." *Journal of Symbolic Logic* 15 (1950), 81–91.

[5] A. N. Prior. *Objects of Thought*. Edited by P. T. Geach and A. J. P. Kenny. Clarendon Press, 1971.

[6] Timothy Williamson. "Everything." *Philosophical Perspectives* 17 (2003), 415–465.