

The contingency of CH

General forcing lemmas and two applications

Proof draft for Andrew Bacon

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The following is a write up of a note “Mathematical Possibilism” dated June 17 2024, sketching a proof of the contingency of the continuum hypothesis in Classicism from the principle Mathematical Possibilism, $\Box$ Rigid Comprehension, Intensional Choice, and $\Box$ Infinity.

The following was written up by ChatGPT Astra, using the essential structure of the proof I gave there, but with a correction to the statement of Mathematical Possibilism noted below. I will eventually write the argument up myself at some point.

A couple of things to note. In *Could the truths of mathematics have been different?* I stated the following theorem, but incorrectly omitted the principle $\Box$ Infinity in the statement of the theorem, which was stated and needed in the original note. While one can derive the axiom of infinity at some suitable type from Mathematical Possibilism, one cannot derive its necessity unless we assume $\Box$ Mathematical Possibilism. The notion of ‘smallness’ used in the statement of Mathematical Possibilism, as presented in the paper and in the original note, is too weak.¹ In this note I have interpreted *Small* to mean ‘bounded by the first strong inaccessible’, although other notions of smallness can be substituted here that would do the same work.

1 Assumptions and conventions

Work in Classicism. Types are generated from e, t by $\rightarrow$; all type indices below are schematic, not object-language variables. Write $\vec{\sigma} \rightarrow t$ for $\sigma_1 \rightarrow \cdots \rightarrow \sigma_n \rightarrow t$. Put $\Box p \equiv (p =_t \top)$, $\Diamond p \equiv \neg \Box \neg p$, $p \leq q \equiv \Box(p \rightarrow q)$, and $X \leq Y \equiv \Box \forall \vec{x}(X\vec{x} \rightarrow Y\vec{x})$. Current coextension is $X \approx Y \equiv \forall \vec{x}(X\vec{x} \leftrightarrow Y\vec{x})$. Appendix ?? expands every nonlogical abbreviation in the following assumptions. In particular maps are relations; function terms and ordinary functional choice are not assumed.

1. Necessary Rigid Comprehension. For each finite nonempty list of argument types $\vec{\sigma}$, assume

$$\boxed{\Box \forall F^{\vec{\sigma} \rightarrow t} \exists X^{\vec{\sigma} \rightarrow t} (\text{Rig}_{\vec{\sigma}} X \wedge X \approx F)}. \quad (\Box \text{RC})$$

Here rigidity is strong rigidity (persistence and inextensibility), not merely persistence.

2. Intensional Choice. For each type σ , assume

$$\boxed{\forall F^{\sigma \rightarrow t} [\Box \exists x^\sigma Fx \rightarrow \exists G^{\sigma \rightarrow t} (G \leq F \wedge \Box \exists ! x^\sigma Gx)]}. \quad (\text{IC})$$

IC itself is not boxed. Only its displayed modal subformulas are boxed.

¹Thanks to Christopher Sun for pointing this out: the rough reason is that although the axiom implies there are infinitely many propositions, it does not imply that there are uncountably many propositions, because there are no countable complete Boolean algebras.

3. Mathematical Possibilism. Fix a type χ , a carrier $K : \chi \rightarrow t$, and a strict order $L : \chi \rightarrow \chi \rightarrow t$ satisfying

$$\text{Inacc}_\chi(K, L). \quad (\text{Size premise})$$

This is the assertion that (K, L) presents a strongly inaccessible cardinal, defined explicitly in Appendix ???. Write κ for the current cardinal it presents; this is notation, not a further object-language constant. Its existence is part of the assumptions, not a consequence of the definition of Small. For $a \in K$, put $K_{<a} = \lambda x.(Kx \wedge Lxa)$, and define

$$\text{Small}_{K,L}^\sigma(B) \equiv \exists a^\chi (Ka \wedge B \preccurlyeq K_{<a}).$$

Thus Small means cardinality *strictly below* the fixed strongly inaccessible cardinal. For every carrier type σ , assume

$$\boxed{\forall B^{\sigma \rightarrow t} \forall E^{\sigma \rightarrow \sigma \rightarrow t} [\text{CBA}_\sigma(B, E) \wedge \text{Small}_{K,L}^\sigma(B) \rightarrow \exists P^t \exists H^{\sigma \rightarrow t \rightarrow t} \text{Iso}(H; B, E; J_P, E_t)]}. \quad (\text{MP})$$

Here $J_P = \lambda q^t. \Box(q \rightarrow P)$ and $E_t = \lambda p^t q^t. \Box(p \rightarrow q)$: the target is the full entailment interval $[\perp, P]$. CBA is given in its order presentation, with E a current non-strict order. Isomorphism means a current relational bijection preserving and reflecting that order.

The parameters K, L are shared by all MP instances. Formally one may add these parameters and the displayed assumptions to Classicism; for any finite fragment, the corresponding closed assumption existentially quantifies K, L over the conjunction of the size premise and its MP instances. This does not quantify over types. No necessity of the size premise, of K, L , or of the isomorphism is assumed.

4. Necessary Infinity. Let $\nu = (e \rightarrow t) \rightarrow t$ and define

$$\begin{aligned} 0 &= \lambda X^{e \rightarrow t}. \top, \\ \text{Suc } Q &= \lambda X^{e \rightarrow t}. \exists x^e (Xx \wedge Q(\lambda y^e. (Xy \wedge y \neq x))), \\ N &= \lambda n^\nu. \forall Z^{\nu \rightarrow t} [(Z0 \wedge \forall m^\nu (Zm \rightarrow Z(\text{Suc } m))) \rightarrow Zn]. \end{aligned}$$

Assume

$$\boxed{\Box [\forall n^\nu (Nn \rightarrow 0 \neq \text{Suc } n) \wedge \forall m^\nu n^\nu ((Nm \wedge Nn \wedge \text{Suc } m = \text{Suc } n) \rightarrow m = n)]}. \quad (1)$$

We call (1) $\Box$ Infinity. Successor injectivity is restricted to $N = \text{FinCard}_e$; unrestricted injectivity on all cardinality quantifiers is not intended.

Scope of the assumptions. All comparisons and constructions are current unless explicitly boxed. The inaccessible definition includes the well-orderable strong-limit clause $\mathcal{P}_r(K_{<a}) \preccurlyeq K_{<b}$ for some $b \in K$, for each $a \in K$. This supplies the current powerset bounds used below without assuming a separate global choice principle. Its exact expansion also makes clear why type raising alone would not suffice. The Dedekind-existence proposition in Section 4 is stated without proof, as agreed; the two conclusions are conditional on that deferred result.

2 Rigid representatives and the arithmetic base

Lemma 2.1 (Rigid representatives and restricted Barcan). *Under RC , every current relation has a unique coextensive rigid relation. If A is rigid, then*

$$\forall x \in A \Box \varphi(x) \implies \Box \forall x \in A \varphi(x), \quad (2)$$

$$\Diamond \exists x \in A \varphi(x) \implies \exists x \in A \Diamond \varphi(x). \quad (3)$$

The same statements hold for rigid relations with several arguments.

Proof. The first implication is inextensibility, applied to the property $\lambda x.\varphi(x)$. The second is its classical contrapositive. For uniqueness, suppose rigid A, B are coextensive. Every current A -member is a B -member and hence necessarily a B -member, by persistence of B . Inextensibility of A gives $A \leq B$; symmetrically $B \leq A$. Intensionality gives $A = B$. $\square$

Write $X^\dagger$ for this unique rigid representative. This is initially a relational abbreviation for unique existence, not an assumed function term taking X to $X^\dagger$. In particular, it must not be used to smuggle in unrestricted Functional Plenitude.

Lemma 2.2 (The natural-number domain). *Under $\Box RC + \Box \text{Infinity}$, N is rigid, carries the usual second-order Peano arithmetic, and satisfies*

$$\forall m, n \in N (m \neq n \rightarrow \Box(m \neq n)).$$

The Peano laws hold necessarily.

Proof. Induction follows from the definition of N . Induction applied to $\lambda n.\Box Nn$ proves $Nn \rightarrow \Box Nn$: $N0$ is a theorem, and successor closure is a theorem, so both may be necessitated.

Choose a rigid $A \approx N$. For each current $n \in A$, $\Box Nn$; thus $\Box(A \subseteq N)$ by (2). Also $A0$ and hence $\Box A0$. For every current $n \in A$, $A(\text{Suc } n)$, and persistence gives $\Box A(\text{Suc } n)$. Inextensibility now gives necessary successor closure of A . At every possibility, induction for N therefore gives $N \subseteq A$. Thus $N = A$ by Intensionality, proving rigidity.

For ND, use induction on n with induction property $\forall m \in N (n \neq m \rightarrow \Box(n \neq m))$. Every natural is 0 or a successor of a natural, by induction. For $n = 0$, a distinct m is a successor, and the necessary zero law applies. For $n = \text{Suc } k$, the case $m = 0$ again uses that law. Otherwise write $m = \text{Suc } l$. If $\text{Suc } k \neq \text{Suc } l$, then $k \neq l$; the induction hypothesis gives $\Box(k \neq l)$ and necessary successor injectivity gives $\Box(\text{Suc } k \neq \text{Suc } l)$. Membership in N persists, so the restricted necessary laws apply to these parameters.

Addition, multiplication, finite sequences and the natural order are defined by the usual relational recursions on this Peano domain. The standard second-order proofs establish their arithmetic laws and that the natural order is a well-order. Applying the same proofs under the boxed hypotheses gives their necessary versions. Recursion graphs, rather than an unrestricted choice of function objects, suffice. $\square$

For rigid A put

$$\mathcal{P}_r(A) = \lambda X.(\text{Rig } X \wedge X \subseteq A).$$

Its members are the unique rigid representatives of current subsets of A . The collection $\mathcal{P}_r(A)$ itself is not assumed rigid.

3 Inductive order types and the current Hartogs construction

For a carrier type τ , put $\rho_\tau = (\tau \rightarrow \tau \rightarrow t) \rightarrow (\tau \rightarrow t) \rightarrow t$. A well-order is given by a strict relation R and its specified field F . The separate field argument includes empty and singleton orders without ambiguity. One may instead encode fields by the diagonals of reflexive well-order relations, obtaining the shorter type $(\tau \rightarrow \tau \rightarrow t) \rightarrow t$.

Use the inductive definitions in the book draft:

$$\begin{aligned} 0_\tau(R, F) &\equiv \text{WO}(R, F), \\ (\text{Suc } \alpha)(R, F) &\equiv \text{WO}(R, F) \wedge \exists \text{ proper initial segment } (S, G) \text{ of } (R, F) \ \alpha(S, G), \\ (\text{Lim } Y)(R, F) &\equiv \text{WO}(R, F) \wedge \forall \beta (Y\beta \rightarrow \beta(R, F)), \\ \alpha!(R, F) &\equiv \alpha(R, F) \wedge \neg(\text{Suc } \alpha)(R, F). \end{aligned}$$

In the limit closure clause, Y is rigid. The property OT_τ is the least collection containing 0_τ , closed under successor, and closed under limits of rigid subcollections. The guard $\text{WO}(R, F)$ in the successor definition is explicit here. These ordinal successors are distinct from the cardinality-quantifier successor in (1).

Lemma 3.1 (Structurality). *Every inductive order type is necessarily invariant under isomorphism of well-orders and is necessarily upward closed under initial-segment embedding. In particular, $\text{Suc } \alpha \leq \alpha$ for $\text{OT}(\alpha)$.*

Proof. Apply order-type induction to the property of necessarily having these two features. The zero and successor cases follow by transporting initial segments along isomorphisms and inclusions. In the limit case, the induction hypothesis supplies the necessary features for every current member of the rigid collection Y . Restricted Barcan makes them available for every Y -member at every possibility. Intersection with the well-orders then preserves both features. $\square$

Lemma 3.2 (Ranks and normalization). *Let $W = (R, F)$ be a current well-order on type τ . Include the whole order among its initial segments. There is a relational assignment of inductive rank properties r_i to these segments such that:*

$$(i) \ r_0 = 0_\tau \text{ and} \qquad r_i = \text{Lim}(\{\text{Suc } r_j : j < i\}^\dagger). \qquad (R)$$

(ii) *For initial segments W_j , $r_i(W_j)$ iff $i \leq j$, and $r_i!(W_j)$ iff $i = j$.*

(iii) *If $\text{OT}(\alpha)$ and $\alpha(W_j)$, then $\alpha = r_i$ for some $i \leq j$. This is identity of properties, not just coextension.*

The subscripts denote positions in this current well-order, not previously available ordinal objects. Its terminal position can be handled separately, so no extra individual or product type is required.

Proof. Use relational well-founded recursion to define (R). At a stage the previous values determine a property of their successors; RC gives its unique rigid representative, and Lim gives a unique next value. The usual recursion proof uses the union of all compatible solutions on initial segments. Two such solutions agree by a least-disagreement argument; if their union stopped short, the displayed rule would extend it by one stage. This uses unique existence of the next value, not a function term selecting rigidifications or ordinary Choice. Induction on the current order proves that every value is an order type.

For $j < i$, persistence of membership in the collection defining r_i gives $r_i \leq \text{Suc } r_j \leq r_j$. At a successor position the strongest member of that collection is $\text{Suc } r_i$, whence $r_{i+1} = \text{Suc } r_i$: the two defining intersections necessarily agree, by inextensibility. Induction on i proves (ii). Indeed, $(\text{Suc } r_k)(W_j)$ iff some proper initial segment W_l of W_j satisfies r_k , iff $k < j$. Consequently (R) holds of W_j exactly when every $k < i$ is below j , that is, when $i \leq j$.

For (iii), use order-type induction on the following property of α : for every current W and every j , if $\alpha(W_j)$ then α is identical to one of its ranks through j . The zero case is immediate. In the successor case, a witnessing proper segment and the induction hypothesis identify the predecessor with r_i for $i < j$; the successor identity just proved gives r_{i+1} .

For the limit case let $\alpha = \text{Lim } Y$, with Y rigid, and suppose $\alpha(W_j)$. Every $\beta \in Y$ is, by induction, identical to a rank r_i with $i \leq j$. Let s be the supremum of these positions in the collection of initial segments of W_j , allowing the whole segment and taking the empty supremum to be zero. For every such i , $r_s \leq r_i$. Restricted Barcan for Y gives $r_s \leq \text{Lim } Y$. Conversely, for every $k < s$ there is a $\beta \in Y$ identical to some r_i with $k < i$. Thus

$$\text{Lim } Y \leq \beta = r_i \leq \text{Suc } r_k.$$

Restricted Barcan for the rigid collection in (R) gives $\text{Lim } Y \leq r_s$. Both properties necessarily imply well-orderhood, so the guards cause no discrepancy. Intensionality yields $\text{Lim } Y = r_s$. This also covers an empty Y , for which both sides are 0_τ . This proves (iii). $\square$

Corollary 3.3 (Representation and comparison). *Every current well-order has exactly one exact inductive order type. Two current well-orders on type τ have the same exact type iff they are isomorphic. The realized types of their proper initial segments are ordered in precisely the same way as their positions.*

Proof. The terminal rank supplies existence. Normalization and (ii) supply uniqueness. An isomorphism identifies corresponding ranks by recursion: the preceding successor-rank collections have the same current members, so their rigid representatives are identical. Conversely, comparison of well-orders gives either an isomorphism or an isomorphism of one with a proper initial segment of the other. The latter alternative gives distinct ranks by (ii), so cannot yield the same exact type. $\square$

Lemma 3.4 (Typed Hartogs construction). *For a collection A at type τ , let*

$$H(A) = \lambda\alpha^{\rho_\tau}. \text{OT}(\alpha) \wedge \exists R, F (F \subseteq A \wedge \text{WO}(R, F) \wedge \alpha!(R, F)).$$

Ordered by proper-initial-segment comparison, $H(A)$ is currently well-ordered, does not inject into A , and every proper initial segment of $H(A)$ does inject into A .

Proof. Corollary 3.3 makes comparison independent of the chosen realizing orders. The usual comparison theorem for well-orders gives totality. To find the least member of a nonempty subcollection, choose one member α and a realizing order W on a subcollection of A . The types below α correspond exactly to the proper initial segments of W . If the subcollection has a member there, take its least corresponding position in W ; otherwise α is least. This is one existential choice in a proof, not a choice function.

The predecessors of each $\alpha \in H(A)$ are order-isomorphic to a realizing order for α , and hence inject into A . If $H(A)$ itself injected into A , transport its well-order along that injection. The transported order would have some exact type $\beta \in H(A)$. Its predecessor collection would therefore be isomorphic to all of $H(A)$, making a well-order isomorphic to a proper initial segment of itself, which is impossible. $\square$

Apply this construction to N and then to $H(N)$. It gives current well-orders of types ω_1 and ω_2 , without ordinary Choice. These two collections initially have different carrier types. In the second one, take the initial segment isomorphic to the first. Call the resulting collections $I_1 \subseteq I_2$, so that

$$|I_1| = \aleph_1, \quad |I_2| = \aleph_2. \quad (4)$$

Both now have the same carrier type. Current cardinal arithmetic for well-ordered sets gives

$$N \times N \cong N, \quad I_1 \times N \cong I_1, \quad I_2^{<\omega} \cong I_2. \quad (5)$$

All three comparisons in (5) are current comparisons, not claims of modal necessity. They are the usual finite-sequence and well-ordered-cardinal arithmetic arguments. In particular, no well-ordering of an arbitrary powerset is used. None of the results of this section asserts ND for I_1 or I_2 .

4 Real-number structures

Proposition 4.1 (Dedekind construction; stated without proof). *In the present setting, $\Box$ Infinity, with $\Box$ RC as background, implies*

$$\Box \exists \Re \text{Real}_\eta(\Re)$$

at a suitably high fixed type η .

The intended construction codes the integers and rationals from N and represents real numbers by rigid representatives of Dedekind cuts. This is the typed adaptation of Dedekind's construction [1]; its verification in Classicism is deferred as requested. The assertion is $\Box \exists \Re \text{Real}(\Re)$, not $\exists \Re \Box \text{Real}(\Re)$.

Lemma 4.2 (Transfer of the powerset formulation). *At a possibility where a real-number structure exists, CH holds if $\mathcal{P}_r(N)$ is well-orderable of cardinality $\aleph_1$. Conversely, suppose $G \subseteq \mathcal{P}_r(N)$, G does not inject into N , and $\mathcal{P}_r(N)$ does not inject into G . Then that real-number structure fails CH.*

Proof. There is an injection $j : \mathcal{P}_r(N) \hookrightarrow \mathbb{R}$, using ternary expansions with digits 0, 2, and an injection $k : \mathbb{R} \hookrightarrow \mathcal{P}_r(N)$, using rational lower cuts and a coding of rationals by naturals. Completeness gives the values of the ternary series; the first differing digit and the tail estimate give injectivity. Density of the rationals gives injectivity of the lower-cut map. Rigid representatives make extensions unique, by Lemma 2.1. These are relational injections and require no Choice. The Cantor–Bernstein theorem gives equinumerosity of $\mathcal{P}_r(N)$ and the real-number domain. If $\mathcal{P}_r(N)$ has well-ordered cardinality $\aleph_1$, every uncountable subcollection of it, and hence of the reals, has that same cardinality. This proves the positive assertion.

The subcollection $j[G]$ is uncountable. If it were equinumerous with $\mathbb{R}$, the injection j followed by that bijection and the inverse of $j|_G$ would give $\mathcal{P}_r(N) \hookrightarrow G$, contrary to hypothesis. Thus $j[G]$ violates the subset formulation of CH. The same constructions work in any complete Archimedean ordered-field structure on N 's arithmetic base. $\square$

5 Stable presentations of countable order types

This section is needed only for the positive application. The inductive definition of order type remains the one in Section 3. We construct an auxiliary presentation of its current countable types, with stronger modal behavior than was required of I_1, I_2 .

Lemma 5.1 (Transport over rigid ND domains). *Let A be rigid and satisfy ND. Rigidification transports a current relation on A to a relation with exactly the same positive and negative atomic facts about current members at every possibility. Current first-order relational laws on A , including functionality, linearity, and injectivity, consequently persist. A rigidification of a current well-order on A is necessarily a well-order. Distinct rigid subcollections of A are necessarily distinct.*

Proof. Positive facts persist. For a tuple outside a relation, each current tuple inside it differs at some coordinate; ND makes that difference necessary. Inextensibility then excludes the outside tuple necessarily. Restricted Barcan for the finitely many copies of A transports the first-order laws. The same argument works for products of rigid ND domains and their rigid subcollections.

For well-foundedness, let R be the rigidified order. Write $B(x)$ for the assertion that its segment through x is not well-founded. In any linear order, $B(x)$ implies $\exists y (Ryx \wedge B(y))$: from a nonempty subset with no least member, first find a member below x , then restrict to the segment through that member. If some $B(x)$ is possible, restricted Barcan extracts a current such x . Take the least current x for which $\Diamond B(x)$. The displayed implication and restricted Barcan for R extract a current predecessor y with $\Diamond B(y)$, a contradiction. A possible failure of well-foundedness of the whole order would give a possible bad segment, by the same no-least-member argument. Thus R is necessarily well-founded.

Finally, distinct rigid subsets have a current witness in their symmetric difference. Its positive and negative memberships persist by the first part, so the subsets remain distinct. $\square$

Lemma 5.2 (A stable countable-segment index). *There are a rigid collection A satisfying ND and a rigid relation $<$ such that currently $(A, <)$ has order type ω_1 , necessarily it is a well-order, and necessarily each of its proper initial segments injects into N . No ND assertion about all the original inductive ordinal properties is needed.*

Proof. Code a well-order on a subset of N by a rigid binary relation on N , using its diagonal for its field and its off-diagonal part for the strict order. Let C be the rigidification of the collection of all current such codes. Lemma 5.1 shows that every current code remains a well-order; restricted Barcan gives this for every member of C necessarily. Distinct codes are necessarily distinct, since they are rigid subsets of N^2 . Thus C is a rigid ND domain.

For current codes, isomorphism and proper-initial-segment comparison persist: rigidify a witnessing map on N^2 and apply transport. Their negations persist as well, by current comparison of well-orders and the impossibility of a well-order being isomorphic to a proper initial segment of itself. Hence the comparison relation on current codes cannot change, even if other codes exist at another possibility.

For each current isomorphism class of codes in C , take its unique rigid representative as a subcollection of C . Let A be the rigid collection of these representatives. Distinct members of A are necessarily distinct by Lemma 5.1 applied to C . Rigidify the current strict comparison relation between these classes to obtain $<$. Currently $(A, <)$ is isomorphic to $H(N)$: associate to a class its unique exact inductive type, using Corollary 3.3. Thus it has order type ω_1 . Transport now makes $<$ necessarily a well-order on A .

For each current $a \in A$, its predecessor segment currently injects into N . Rigidify the graph of such an injection. Transport over A and N makes it necessarily an injection from the predecessor segment of a into N . Restricted Barcan for A gives the asserted necessary statement for all its members. Choosing a witness separately for a fixed a in this proof requires no simultaneous choice of maps. $\square$

The class representatives here are auxiliary codes, not a replacement definition of the inductive ordinals. The lemma proves stability for this presentation; it does not infer necessary identity

between every presentation of an ordinal. It also does not yet assert that A remains uncountable. The positive application will establish that on its P .

6 General Boolean-algebra and naming lemmas

Lemma 6.1 (Regular-open completion). *For a current nonempty partial order $\mathbb{Q}$, the rigid representatives of its regular open subsets form a current nontrivial CBA $\text{RO}(\mathbb{Q})$. Its size is at most $|\mathcal{P}_r(\mathbb{Q}^\dagger)|$. For the two partial-function orders below, every basic cone is regular open and every nonzero regular open contains a cone.*

Proof. For a downward closed V , set

$$j(V) = \{c : \forall d \leq c \exists f \leq d (f \in V)\}.$$

The fixed points of j are the regular opens. Arbitrary joins are $j(\bigcup_i U_i)$, finite meets are intersections, and the complement of U consists of conditions with no extension in U . These give the complete Boolean algebra after taking unique rigid representatives. All these claims are current extensional facts about the partial order. Distinct rigid representatives have different current extensions, giving the size bound. A nonempty open contains a condition and its cone. In either partial-function order, a condition not extending c has an extension incompatible with c : disagree on an unset coordinate of c , unless there is already a disagreement. Hence the cone of c is regular open. The empty condition witnesses nontriviality. $\square$

For any application, MP supplies a current relational isomorphism

$$h : \mathbb{B} \cong [\perp, P], \quad P \neq \perp. \quad (6)$$

Its current unique outputs are denoted $h(b)$; this notation makes no claim of modal functionality. As an order isomorphism it preserves all existing joins and meets, and relative complements: $h(\neg b) = P \wedge \neg h(b)$.

Lemma 6.2 (Density to entailment). *If, below each nonzero $b \in \mathbb{B}$, there is a nonzero $c \leq b$ with $h(c) \leq \varphi$, then $P \leq \varphi$. The same statement holds below any fixed nonzero element of $\mathbb{B}$.*

Proof. Otherwise the preimage of $P \wedge \neg \varphi$ is nonzero; a refinement as stipulated would have image entailing a contradiction. $\square$

Lemma 6.3 (Partial functional representation). *Suppose a current relation $T(n, p)$ is total and functional from N to propositions. Then there is a unique $X : \nu \rightarrow t$ such that*

$$\Box(X \subseteq N), \quad \forall n \in N \forall p (T(n, p) \rightarrow Xn = p).$$

Proof. If $T(n, p)$ and $T(m, q)$ currently hold, then $\Box(n = m \rightarrow p = q)$. For $n = m$ use functionality and necessity of identity; for $n \neq m$ use ND for N . Rigidify T to $T^\dagger$ and define

$$f(n) = \exists p (T^\dagger(n, p) \wedge p), \quad Xn = Nn \wedge f(n).$$

For a current pair $T(n, p)$, persistence gives $p \leq f(n)$. The boxed conditional functionality above and restricted Barcan for $T^\dagger$ give $\Box \forall q (T^\dagger(n, q) \rightarrow q = p)$; hence $f(n) \leq p$. Intensionality gives $f(n) = p$. Since $n \in N$ persists, $Nn = \top$, so $Xn = p$ as required.

If X, X' both satisfy the requirements, $Xn = X'n$ for each current $n \in N$. Necessity of identity and restricted Barcan for N , together with necessary inclusion in N , give necessary coextension everywhere. Thus $X = X'$. $\square$

Lemma 6.4 (Local selection). *Let $\Phi(I)$ be invariant under current coextension of the relation I , with that invariance necessary. If $Q \leq \exists I \Phi(I)$, IC supplies a current relation I such that $Q \leq \Phi(I)$.*

Proof. Necessarily some relation satisfies $Q \rightarrow \Phi$: when Q holds use the hypothesis, and otherwise use any relation. Apply IC to that property, obtaining V with $\Box \exists! J V J$ and $V \leq \lambda J. (Q \rightarrow \Phi(J))$. Define $I(\vec{x}) = \exists J (V J \wedge J(\vec{x}))$. Necessarily I is coextensive with the unique selected J , so the assumed invariance gives $Q \leq \Phi(I)$. $\square$

Lemma 6.5 (Counting serial relations). *Let $J \subseteq A \times B$ be current and serial, with B currently well-ordered. Suppose each fibre over b is uniformly injected into N . Then $A \hookrightarrow B \times N$.*

Proof. Send a to its least J -related b , together with its fibre code. This is a uniquely specified relation and is injective. $\square$

Definition 6.6 (Countable binary decision property). A current CBA $\mathbb{B}$ has this property if, for every current relational sequence $(b_n)_{n \in N}$ in $\mathbb{B}$ and nonzero b , there is a nonzero $c \leq b$ such that for every n , either $c \leq b_n$ or $c \leq \neg b_n$. This is the form of countable distributivity we use.

Lemma 6.7 (Closure supplies simultaneous decisions). *Suppose $\mathbb{Q}$ is currently well-orderable and every descending N -sequence has a lower bound. Then $\text{RO}(\mathbb{Q})$ has the countable binary decision property, provided basic cones form a dense family of nonzero Boolean elements (as in our two applications).*

Proof. Start with a cone below the prescribed nonzero b . Given c_n , choose the least extension c_{n+1} whose cone lies below b_n or its Boolean complement. Such extensions are dense: if the cone meets b_n , refine into the intersection; otherwise it lies in the complement. Relational recursion on N gives the descending sequence, and a lower bound has a nonzero cone deciding every b_n . Only the fixed well-order is used for selection; no appeal to IC as dependent choice is made. $\square$

Lemma 6.8 (No new rigid subsets). *Suppose $\mathbb{B}$ has the countable binary decision property and $h : \mathbb{B} \cong [\perp, P]$. Let $R_0 = (\mathcal{P}_r(N))^\dagger$, the rigidification of the current collection of rigid subsets of N . Then*

$$P \leq \forall U [\text{Rig } U \wedge U \subseteq N \longleftrightarrow R_0 U].$$

Proof. First fix a current property X and $0 \neq Q \leq P$. Pull back the propositions $P \wedge Xn$ to a current sequence in $\mathbb{B}$. Simultaneous decision below the preimage of Q gives $0 \neq q \leq Q$ deciding every Xn . Let S be the rigid representative of $\{n \in N : q \leq Xn\}$. Its memberships and nonmemberships among current naturals persist by Lemma 5.1. Restricted Barcan for N therefore gives

$$q \leq \forall n \in N (Xn \leftrightarrow Sn). \tag{D}$$

Every current member of R_0 is necessarily a rigid subset of N : rigidity is itself a necessary property in the present S5 setting, and its current inclusion in rigid N persists by restricted Barcan. Restricted Barcan for R_0 proves one direction of the lemma.

If the other direction failed, some $0 \neq Q \leq P$ would entail the existence of a rigid subset of N outside R_0 . Apply local selection to the coextension-invariant property of X

$$\exists U (\text{Rig } U \wedge U \subseteq N \wedge U \approx X \wedge \neg R_0 U).$$

This gives a current X satisfying that property throughout Q . Apply (D) below Q . Since Q entails $X \subseteq N$, the resulting q entails $X \approx S$. Its witness U is rigid and coextensive with rigid S , so $U = S$. But S is a current member of R_0 and that membership persists, a contradiction. Notice that local selection was applied to an extensional property of X , not to “ X is rigid”, which is not invariant under current coextension. $\square$

7 Application I: finite conditions and possible failure of CH

Let $\mathbb{C}$ be the current partial order of finite partial functions $I_2 \times N \rightarrow 2$, ordered by extension: stronger conditions extend weaker ones. Graphs can be represented by rigid relations, while finiteness and functionality in this definition are current conditions. No graph is assumed necessarily functional. Fix a current well-order of $\mathbb{C}$, using finite-sequence coding and (5).

Let $\mathbb{B} = \text{RO}(\mathbb{C})$. Lemma 6.1 gives completeness. Finite-sequence coding injects $\mathbb{C}$ into I_2 , so $|\mathbb{B}| \leq 2^{\aleph_2} < \kappa$ under the size convention above. Thus MP supplies (6) for this application.

Lemma 7.1 (Uniform CCC). *Every antichain of nonzero elements of $\mathbb{B}$ has a canonical injection into N , relative to the fixed well-order of $\mathbb{C}$. The construction is uniform in a parameter indexing a family of antichains.*

Proof. For each U take its least condition c_U . Different members of an antichain give incompatible conditions. Fix m . There are at most 2^m pairwise incompatible conditions with domain size m . To see this, take any finite family and let d be the union of its domains. Each condition extends to exactly $2^{|d|-m}$ total assignments $d \rightarrow 2$; these sets of assignments are disjoint. Finite counting gives at most 2^m conditions. A larger collection would contain a finite subfamily of size $2^m + 1$, so the same bound holds for the whole group.

Assign U the number $2^m + r$, where $r < 2^m$ is the rank of c_U among the selected conditions of size m in their fixed order. These codes are distinct. Least-condition selection and finite rank are unique definitions, so no countable choice is needed, even for a family of antichains. $\square$

For $a \in I_2$ and $n \in N$, let b_{an} be the regular open consisting of conditions assigning 1 to (a, n) , and let p_{an} be its output under (6). Holding a fixed, apply Lemma 6.3 to the relation between n and p_{an} . Define

$$\text{Name}(a, X) \iff \Box(X \subseteq N) \wedge \forall n \in N (Xn = p_{an}).$$

Then $\forall a \in I_2 \exists! X \text{ Name}(a, X)$. This is a current functional relation. We do *not* infer the existence of a function term $a \mapsto X$, and do not assume ND for a .

Lemma 7.2 (Cohen separation). *If $a \neq a'$ are current indices and $\text{Name}(a, X)$, $\text{Name}(a', X')$, then*

$$P \leq \neg(X \approx X').$$

Consequently distinct indices have distinct current names.

Proof. Let $0 \neq q \leq P$ and take its corresponding nonzero regular open. Choose a condition c in it. Choose a natural n fresh for both coordinates (a, n) and (a', n) in the finite domain of c . Extend c to d assigning 1 to the first coordinate and 0 to the second. The image q' of the nonempty cone $[d]$ satisfies

$$0 \neq q' \leq q, \quad q' \leq p_{an} \wedge \neg p_{a'n} = Xn \wedge \neg X'n.$$

The equalities specifying each name are propositional identities, so this computation is valid under entailment. Taking $q = P \wedge (X \approx X')$ shows that this proposition cannot be nonzero: otherwise q' would entail both coextension and its negation. If $X = X'$ currently, the displayed conclusion would contradict $P \neq \perp$. Thus the assignment of indices to names is injective. $\square$

Let D be the rigid representative of the current collection of names for indices in I_2 , and E the rigid representative of those for I_1 . Current unique existence and injectivity give

$$|D| = \aleph_2, \quad |E| = \aleph_1. \tag{7}$$

Transport the current index well-orders to D, E when needed for counting. Their members necessarily are subsets of N , by restricted Barcan. For any distinct current $X, X' \in D$, Lemma 7.2 applies. Nothing requires those names or their original indices to remain distinct outside P .

Define the intended intermediate collection

$$G(Y) \iff \text{Rig } Y \wedge Y \subseteq N \wedge \exists X \in E (Y \approx X). \quad (8)$$

At every possibility each $X \in E$ has exactly one rigid representative of its extension, by $\square\text{RC}$. That representative belongs to G . The definition uses the rigid collection E , not the formula “is currently a name of an index below ω_1 ” re-evaluated at other possibilities.

Lemma 7.3 (The intermediate collection is uncountable on P). $P \leq \neg \exists I (I : G \hookrightarrow N)$.

Proof. Otherwise $Q = P \wedge \exists I (I : G \hookrightarrow N)$ is nonzero. Lemma 6.4 fixes I with $Q \leq (I : G \hookrightarrow N)$. For current $X \in E$ and $n \in N$ put

$$d_{Xn} = Q \wedge \exists U (\text{Rig } U \wedge U \approx X \wedge IU n), \quad J(X, n) \iff d_{Xn} \neq \perp.$$

For each current X , throughout Q the rigid representative of X is in G and is mapped to some $n \in N$. Since Q is possible, (3) for N gives a current n with $d_{Xn} \neq \perp$. Thus J is serial.

For fixed n , distinct current $X, X' \in E$ give incompatible $d_{Xn}, d_{X'n}$. If both held, injectivity of I would identify their rigid representatives, making $X \approx X'$, contrary to separation on P . Pulling the nonzero d_{Xn} back through h therefore gives an antichain in $\mathbb{B}$. Uniform CCC and Lemma 6.5 yield $E \hookrightarrow N \times N$, contradicting (7) and (5). $\square$

Lemma 7.4 (The continuum does not inject into G on P). $P \leq \neg \exists I (I : \mathcal{P}_r(N) \hookrightarrow G)$.

Proof. Otherwise let $Q = P \wedge \exists I (I : \mathcal{P}_r(N) \hookrightarrow G) \neq \perp$, and choose I valid throughout Q by local selection. For current $X \in D$ and $Y \in E$ put

$$d_{XY} = Q \wedge \exists U, V (\text{Rig } U \wedge \text{Rig } V \wedge U \approx X \wedge V \approx Y \wedge IUV), \\ J(X, Y) \iff d_{XY} \neq \perp.$$

Throughout Q , the rigid extension of X belongs to $\mathcal{P}_r(N)$, so its image under I is in G . Definition (8) supplies some $Y \in E$ with the required extension. Possibility and (3) for E give a current such Y with $d_{XY} \neq \perp$. Hence J is serial.

For fixed current Y , the nonzero d_{XY} are pairwise incompatible as X ranges over distinct current members of D . Two simultaneous witnesses have rigid outputs coextensive with Y , hence identical by Lemma 2.1. Injectivity of I identifies the inputs, contradicting separation of the corresponding X 's on P . The pullbacks form an antichain in $\mathbb{B}$. Uniform CCC and the counting lemma give $D \hookrightarrow E \times N$. But $|D| = \aleph_2$ whereas $|E \times N| = \aleph_1$, a contradiction. $\square$

Theorem 7.5 (Theorem 3.1: possible failure of CH). *Assume Classicism, $\square\text{RC}$, IC, $\square\text{Infinity}$, and MP with size coverage for the Cohen algebra of Section 7. Subject to Proposition 4.1, stated without proof, at a sufficiently high fixed type η ,*

$$\Diamond \exists \mathfrak{R} (\text{Real}_\eta(\mathfrak{R}) \wedge \neg \text{CH}(\mathfrak{R})).$$

Proof. The constructions above give $P \neq \perp$ and $G \subseteq \mathcal{P}_r(N)$. Lemmas 7.3 and 7.4 show that throughout P , G neither injects into N nor admits an injection from $\mathcal{P}_r(N)$. Real-number structures necessarily exist by Proposition 4.1. At every P possibility, Lemma 4.2 therefore supplies a real-number structure failing CH. Since P is possible, the displayed conclusion follows. $\square$

8 Application II: countable conditions and possible CH

Use the rigid index order $(A, <)$ of Lemma 5.2, and let $\mathbb{Q}$ consist of current partial functions

$$c : A \times N \rightarrow 2 \quad \text{with domain injecting into } N,$$

ordered by extension. Graphs are represented rigidly. The reference to countability in this definition is current. Set $\mathbb{B}_+ = \text{RO}(\mathbb{Q})$.

Lemma 8.1 (Closure, size, and MP coverage). *Under the size convention in Section 1, $\mathbb{Q}$ is well-orderable and countably closed. The algebra $\mathbb{B}_+$ has the countable binary decision property and is small.*

Proof. Currently $|A| = \aleph_1$, so every condition is coded by a subset of the well-ordered carrier $A \times N \times 2$, of size $\aleph_1$. The strong-limit convention gives a well-ordering of its powerset, and hence of $\mathbb{Q}$, with size some cardinal $\lambda \leq 2^{\aleph_1} < \kappa$. Its powerset in turn has a well-orderable bound $2^\lambda < \kappa$. Completion therefore gives $|\mathbb{B}_+| \leq 2^\lambda < \kappa$.

For a descending sequence (c_n) , the union of the graphs is a partial function. Its domain is countable as follows. Possible injections of each $\text{dom}(c_n)$ into N are coded by subsets of $A \times N \times N$, another well-ordered carrier of size $\aleph_1$. Fix a well-order of that powerset and select the least such injection for each n . Map a point of the union domain to the least n whose domain contains it and its code under that injection. This injects the union into $N^2 \cong N$. Thus the union is a condition and a lower bound. Lemma 6.7 gives simultaneous decisions. This explicitly accounts for both choices usually hidden in the countable-closure argument: enumerating countable domains and selecting deciding extensions. $\square$

By MP choose $h : \mathbb{B}_+ \cong [\perp, P_+]$ with $P_+ \neq \perp$. In this section h refers to this new isomorphism. For each current $a \in A$, use Lemma 6.3 to construct a unique row name X_a with

$$\Box(X_a \subseteq N), \quad X_a n = h(b_{an}),$$

where b_{an} is the regular open of conditions assigning 1 at (a, n) . Unlike the finite-condition case, no separation of distinct rows is asserted: all current reals will occur among these rows.

Lemma 8.2 (A total row relation). *There is a current relation $L(a, U)$ such that necessarily it is a total functional relation from A into $\mathcal{P}_r(N)$, taking each current a to the rigid representative of the extension of X_a .*

Proof. Rigidify the current name graph $T(a, X)$ to $T^\dagger$. For any two current pairs, ND of A and necessity of identity give

$$\Box(a = a' \rightarrow X = X').$$

Restricted Barcan for $T^\dagger$ therefore makes that graph necessarily functional; persistence and restricted Barcan for A make it necessarily total on A . Its values necessarily are subsets of N , by the name lemma and restricted Barcan for the graph. Define

$$L(a, U) \longleftrightarrow Aa \wedge \text{Rig } U \wedge \exists X (T^\dagger(a, X) \wedge U \approx X).$$

Necessary RC gives existence and uniqueness of the rigid representative at each possibility. This proves the assertion without choosing a function term $a \mapsto X_a$. $\square$

Lemma 8.3 (Every current real occurs in a row). *For every current $S \in \mathcal{P}_r(N)$, $P_+ \leq \exists a \in AL(a, S)$.*

Proof. Below any nonzero Boolean element choose a condition c . Its domain uses only countably many rows: select the least natural coordinate in each used row, and use an injection of $\text{dom}(c)$ into N . Since A is currently uncountable, some current $a \in A$ is unused. Extend c by assigning the whole row a the characteristic function of S . The resulting graph d has countable domain: the original domain and one new countable row can be coded in disjoint parts of N .

The image $q = h([d])$ is nonzero. For every current n , it entails $X_a n \leftrightarrow S n$; memberships and nonmemberships of rigid S persist. Restricted Barcan for N and the necessary subset condition give $q \leq X_a \approx S$. Hence $q \leq L(a, S)$. Lemma 6.2 now gives the result. $\square$

Lemma 8.4 (The row relation enumerates the continuum on P_+). P_+ entails that L is a surjection from A onto $\mathcal{P}_r(N)$.

Proof. Let $R_0 = (\mathcal{P}_r(N))^\dagger$ as in Lemma 6.8. For every current $S \in R_0$, Lemma 8.3 gives $\square(P_+ \rightarrow \exists a \in A L(a, S))$. Restricted Barcan for R_0 gives this universally on R_0 throughout P_+ . Lemmas 8.1 and 6.8 identify R_0 with *all* rigid subsets of N on P_+ . Thus every such subset has a preimage. Totality and functionality are Lemma 8.2. $\square$

Theorem 8.5 (Theorem 3.2: possible CH). *Assume Classicism, $\square\text{RC}$, IC, $\square\text{Infinity}$, and MP with the size and well-orderability package in Section 1. Subject to Proposition 4.1, at the same sufficiently high type η as in Theorem 7.5,*

$$\Diamond \exists \mathfrak{A} (\text{Real}_\eta(\mathfrak{A}) \wedge \text{CH}(\mathfrak{A})).$$

Proof. Work at an arbitrary P_+ possibility. The order $(A, <)$ is a well-order and each proper initial segment is countable, by Lemma 5.2. By Lemma 8.4, L maps A onto $\mathcal{P}_r(N)$. Taking the least preimage in the well-order gives an injection $\mathcal{P}_r(N) \hookrightarrow A$.

Cantor's diagonal argument shows that $\mathcal{P}_r(N)$ does not inject into N : an injection would give a surjection from N onto $\mathcal{P}_r(N)$ by inverting it on its image and using the empty subset elsewhere, contradicting diagonalization. Hence A is uncountable. A well-order that is uncountable and has only countable proper initial segments has order type ω_1 . Thus $\mathcal{P}_r(N)$ is an uncountable subcollection of a well-order of type ω_1 , and has cardinality $\aleph_1$.

Proposition 4.1 supplies a real-number structure at this possibility. Lemma 4.2 gives CH for it. Since $P_+ \neq \perp$, this proves the displayed possibility claim. $\square$

9 Dependency summary and scope

Shared machinery	Application-specific work
Rigidification, restricted Barcan, rigid ND naturals	Finite conditions: uniform CCC and separation of rows
Inductive order types and current Hartogs construction	Countable conditions: stable index presentation and closure
Regular-open completion, MP, density to entailment	Negative CH: two serial-relation counting arguments
Separate Boolean names and IC local selection	Positive CH: no new subsets and enumeration by rows
Dedekind existence and powerset transfer	Size bounds for each of the two algebras

Theorem 7.5 uses less than the common sufficient package: the Cohen MP instance suffices once the shared foundations are available. Theorem 8.5, as proved here, additionally uses the current well-orderings needed for closure and simultaneous decisions. Those follow from our explicit well-orderable strong-limit convention, not from IC or type raising alone. Neither proof assumes ND for all order types below ω_2 , nor a necessarily well-ordered domain of that size. The positive proof derives precisely the stable countable-segment presentation it needs.

The two propositions P and P_+ need not be the same: each witnesses one side of the contingency claim. No generic extension, forcing truth lemma, or general theorem transferring arbitrary set-theoretic sentences has been assumed. Only the Boolean facts proved above are transferred through the full entailment intervals supplied by MP.

The necessary Dedekind-existence proposition is deliberately stated without proof. All other displayed lemmas have mathematical proof arguments here; verification against the exact formal axioms of Classicism, and Lean formalization, remain separate tasks.

A Explicit definitions in the language of Classicism

This appendix gives definitional abbreviations, not additional primitive predicates. Each definition can be expanded recursively into typed quantification, application, abstraction, identity and the ordinary propositional connectives. Bound variables in expansions are always chosen fresh to avoid capture. $\Box p$ abbreviates $p =_t \top$. Types written above variables bind those variables at the indicated type; unmarked variables inherit the types stated in their paragraph. Restricted quantifiers abbreviate

$$\forall x \in A \varphi \equiv \forall x(Ax \rightarrow \varphi), \quad \exists x \in A \varphi \equiv \exists x(Ax \wedge \varphi).$$

Also $\exists! x \varphi(x)$ abbreviates $\exists x(\varphi(x) \wedge \forall y(\varphi(y) \rightarrow y = x))$. A displayed tuple is a finite list of parameters, not an object of a new product type. Definitions for predicates on tuples expand into one quantifier for each component. Defined symbols below have implicit type indices when their arguments determine them.

A.1 Rigidity, coextension, and representatives

For $X, Y : \vec{\sigma} \rightarrow t$, define

$$\begin{aligned} X \subseteq Y &\equiv \forall \vec{x}(X\vec{x} \rightarrow Y\vec{x}), \\ X \approx Y &\equiv \forall \vec{x}(X\vec{x} \leftrightarrow Y\vec{x}), \\ X \leq Y &\equiv \Box(X \subseteq Y), \\ \text{Pers}(X) &\equiv \Box \forall \vec{x}(X\vec{x} \rightarrow \Box X\vec{x}), \\ \text{Inext}(X) &\equiv \Box \forall Y^{\vec{\sigma} \rightarrow t}[(\forall \vec{x}(X\vec{x} \rightarrow \Box Y\vec{x})) \rightarrow \Box \forall \vec{x}(X\vec{x} \rightarrow Y\vec{x})], \\ \text{Rig}(X) &\equiv \text{Pers}(X) \wedge \text{Inext}(X). \end{aligned}$$

Equivalently, the rigidity abbreviation in the original note is

$$\Box \forall Y^{\vec{\sigma} \rightarrow t}[\Box \forall \vec{x}(X\vec{x} \rightarrow Y\vec{x}) \leftrightarrow \forall \vec{x}(X\vec{x} \rightarrow \Box Y\vec{x})].$$

Persistence supplies the left-to-right implication; inextensibility supplies the converse. Conversely this biconditional, instantiated with $Y = X$, supplies persistence, and directly supplies inextensibility.

For $A : \sigma \rightarrow t$ define

$$\mathcal{P}_r(A) = \lambda U^{\sigma \rightarrow t}.(\text{Rig } U \wedge U \subseteq A), \quad \text{ND}(A) \equiv \forall x, y \in A (x \neq y \rightarrow \Box(x \neq y)).$$

This definition of $\mathcal{P}_r(A)$ is meaningful even when A is not rigid; it selects rigid representatives of its current subsets. The notation $X^\dagger$ is eliminated relationally: a claim involving it abbreviates a claim about the unique Y satisfying $\text{Rig } Y \wedge Y \approx X$. Its unique existence follows from RC and Lemma 2.1; no choice function $X \mapsto X^\dagger$ is introduced.

A.2 Relations, maps, and cardinal comparisons

Let $A : \sigma \rightarrow t$, $B : \tau \rightarrow t$, and $F : \sigma \rightarrow \tau \rightarrow t$.

$$\begin{aligned} \text{Map}(F; A, B) &\equiv \forall x^\sigma y^\tau (Fxy \rightarrow Ax \wedge By) \\ &\quad \wedge \forall x \in A \exists y \in B Fxy \\ &\quad \wedge \forall x^\sigma y^\tau z^\tau ((Fxy \wedge Fxz) \rightarrow y = z), \\ \text{Inj}(F; A, B) &\equiv \text{Map}(F; A, B) \wedge \forall x^\sigma z^\sigma y^\tau ((Fxy \wedge Fzy) \rightarrow x = z), \\ \text{Surj}(F; A, B) &\equiv \text{Map}(F; A, B) \wedge \forall y \in B \exists x \in A Fxy, \\ \text{Bij}(F; A, B) &\equiv \text{Inj}(F; A, B) \wedge \text{Surj}(F; A, B), \\ A \preccurlyeq B &\equiv \exists F^{\sigma \rightarrow \tau \rightarrow t} \text{Inj}(F; A, B), \\ A \prec B &\equiv (A \preccurlyeq B) \wedge \neg(B \preccurlyeq A), \\ A \simeq B &\equiv \exists F^{\sigma \rightarrow \tau \rightarrow t} \text{Bij}(F; A, B). \end{aligned}$$

Thus $A \hookrightarrow B$ in the proof means $A \preccurlyeq B$; the notation $F : A \hookrightarrow B$ means $\text{Inj}(F; A, B)$. For a multi-place domain the same formulas use a list $\vec{x}$ in place of x , and componentwise identity for the injectivity clause. For example a predicate on $A \times B$ means $\lambda x^\sigma y^\tau. (Ax \wedge By)$, not a new product object. Define $\text{Count}(A) \equiv A \preccurlyeq N$ and $\text{Uncount}(A) \equiv \neg \text{Count}(A)$.

A.3 Orders, bounds, and isomorphism

Let $B : \sigma \rightarrow t$ and $E : \sigma \rightarrow \sigma \rightarrow t$.

$$\begin{aligned} \text{PO}(B, E) &\equiv \forall xy (Exy \rightarrow Bx \wedge By) \wedge \forall x \in B Exx \\ &\quad \wedge \forall x, y \in B ((Exy \wedge Eyx) \rightarrow x = y) \\ &\quad \wedge \forall x, y, z \in B ((Exy \wedge Eyz) \rightarrow Exz). \end{aligned}$$

For $X : \sigma \rightarrow t$ put

$$\begin{aligned} \text{Sup}_{B,E}(X, u) &\equiv Bu \wedge X \subseteq B \wedge \forall x \in X Exu \\ &\quad \wedge \forall v \in B ((\forall x \in X Exv) \rightarrow Euv), \\ \text{Inf}_{B,E}(X, u) &\equiv Bu \wedge X \subseteq B \wedge \forall x \in X Eux \\ &\quad \wedge \forall v \in B ((\forall x \in X Evx) \rightarrow Evu), \\ \text{Join}(x, y, u) &\equiv Bx \wedge By \wedge \text{Sup}_{B,E}(\lambda z. (z = x \vee z = y), u), \\ \text{Meet}(x, y, u) &\equiv Bx \wedge By \wedge \text{Inf}_{B,E}(\lambda z. (z = x \vee z = y), u), \\ \text{Ends}(o, i) &\equiv Bo \wedge Bi \wedge \forall x \in B (Eox \wedge Exi). \end{aligned}$$

In Join, Meet and Ends the fixed parameters B, E are suppressed. For another carrier $C : \tau \rightarrow t$ and relation $D : \tau \rightarrow \tau \rightarrow t$,

$$\begin{aligned} \text{Iso}(H; B, E; C, D) &\equiv \text{Bij}(H; B, C) \\ &\quad \wedge \forall x^\sigma y^\sigma u^\tau v^\tau ((Hxu \wedge Hyv) \rightarrow (Exy \leftrightarrow Duv)). \end{aligned}$$

This definition applies equally to strict and non-strict orders. No box is inserted around the bijection or the preservation condition.

A strict well-order is defined by

$$\begin{aligned} \text{SO}(B, L) &\equiv \forall xy (Lxy \rightarrow Bx \wedge By) \wedge \forall x \in B \neg Lxx \\ &\quad \wedge \forall x, y, z \in B ((Lxy \wedge Lyz) \rightarrow Lxz) \\ &\quad \wedge \forall x, y \in B (x = y \vee Lxy \vee Lyx), \\ \text{WO}(L, B) &\equiv \text{SO}(B, L) \wedge \forall X^{\sigma \rightarrow t} [(X \subseteq B \wedge \exists x Xx) \rightarrow \\ &\quad \exists x (Xx \wedge \forall y (Xy \rightarrow \neg Lyx))]. \end{aligned}$$

Write $B_{<a} = \lambda x. (Bx \wedge Lxa)$ and $L|_X = \lambda xy. (Xx \wedge Xy \wedge Lxy)$. A proper initial segment of (L, B) means $(L|_{B_{<a}}, B_{<a})$ for some $a \in B$. Allowing any isomorphic copy instead is expressed by existentially quantifying a relation satisfying Iso.

A.4 Complete Boolean algebras

Define distributivity using only the preceding relational abbreviations:

$$\begin{aligned} \text{Dist}(B, E) &\equiv \forall x, y, z, u, v, w, d, f \in B [\text{Join}(y, z, u) \wedge \text{Meet}(x, u, v) \\ &\quad \wedge \text{Meet}(x, y, w) \wedge \text{Meet}(x, z, d) \\ &\quad \wedge \text{Join}(w, d, f) \rightarrow v = f]. \end{aligned}$$

Then the required order presentation of CBA is

$$\begin{aligned} \text{CBA}_\sigma(B, E) &\equiv \text{PO}(B, E) \wedge \text{Dist}(B, E) \\ &\quad \wedge \forall X^{\sigma \rightarrow t} (X \subseteq B \rightarrow \exists u^\sigma \text{Sup}_{B, E}(X, u)) \\ &\quad \wedge \exists o^\sigma i^\sigma [\text{Ends}(o, i) \wedge \forall x \in B \exists y \in B (\text{Meet}(x, y, o) \wedge \text{Join}(x, y, i))]. \end{aligned}$$

Arbitrary infima follow by taking the supremum of the lower bounds; thus the displayed completeness condition supplies both joins and meets. The remaining clauses say that the bounded lattice is distributive and complemented. Degenerate Boolean algebras are allowed; nontriviality is $\exists x, y \in B (x \neq y)$. Both applications prove nontriviality. The definitions require no primitive operation symbols for joins, meets or complementation and no actual powerset objects.

For the target of MP, $P : t$, $J_P : t \rightarrow t$ and $E_t : t \rightarrow t \rightarrow t$ are exactly the terms in Section 1. Consequently MP is a typed formula schema after expanding CBA, Small and Iso, with no set-theoretic satisfaction predicate or external reference to Boolean algebras.

A.5 Strong inaccessibility and Small

Fix $K : \chi \rightarrow t$ and $L : \chi \rightarrow \chi \rightarrow t$. All quantifiers in this subsection are object-language quantifiers at the displayed fixed types. Define

$$\begin{aligned} \text{Initial}(K, L) &\equiv \text{WO}(L, K) \wedge \forall a \in K \neg (K \preceq K_{<a}), \\ \text{Regular}(K, L) &\equiv \forall Y^{\chi \rightarrow t} [(Y \subseteq K \wedge Y \prec K) \rightarrow \exists b \in K \forall y \in Y Lyb], \\ \text{StrongLimit}(K, L) &\equiv \forall a \in K \exists b \in K (\mathcal{P}_r(K_{<a}) \preceq K_{<b}), \\ \text{Inacc}_\chi(K, L) &\equiv \text{Initial}(K, L) \wedge N \prec K \\ &\quad \wedge \text{Regular}(K, L) \wedge \text{StrongLimit}(K, L), \\ \text{Small}_{K, L}^\sigma(B) &\equiv \exists a \in K (B \preceq K_{<a}). \end{aligned}$$

Regular says that every subset of K of smaller cardinality is bounded. Initial ensures that (K, L) represents a cardinal rather than just an ordinal of that cardinality. $N \prec K$ says it is uncountable. StrongLimit includes the well-orderability of each indicated powerset. The injection in that clause has type $(\chi \rightarrow t) \rightarrow \chi \rightarrow t$; the injection in Small has type $\sigma \rightarrow \chi \rightarrow t$. No quantifier ranges over types or over untyped sets. In particular “a strongly inaccessible cardinal” here means the fixed witness (K, L) , not an untyped existential ranging over arbitrarily high types. No least-inaccessible requirement is needed.

Under Initial, Small is equivalent to $B \prec K$. One direction follows by composing the injection into a proper initial segment with inclusion. For the other, an injection into K well-orders B by its image. If that image had cardinality $|K|$, it would give $K \preceq B$; otherwise its well-ordered cardinal is represented by a proper initial segment of K . Hence it injects into such a segment.

The strong-limit clause also covers small carriers at other types. Indeed, from $B \preceq K_{<a}$, a witnessing injection sends each current subset of B to a subset of $K_{<a}$; unique rigidification of that image gives a relational injection $\mathcal{P}_r(B) \preceq \mathcal{P}_r(K_{<a})$. StrongLimit then gives $\text{Small}(\mathcal{P}_r(B))$. This argument iterates finitely often and supplies precisely the current well-orderable powerset bounds used in the main proof. It is not an assumption that these bounds persist.

A.6 The inductive definitions and forcing abbreviations

The definition of N and its zero and successor are fully expanded in Section 1. For order types at type $\rho = (\tau \rightarrow \tau \rightarrow t) \rightarrow (\tau \rightarrow t) \rightarrow t$, put

$$\begin{aligned} 0_\tau &= \lambda RF. \text{WO}(R, F), \\ \text{Suc } \alpha &= \lambda RF. (\text{WO}(R, F) \wedge \exists a \in F \alpha(R|_{F_{<a}}, F_{<a})), \\ \text{Lim } Y &= \lambda RF. (\text{WO}(R, F) \wedge \forall \beta^\rho (Y\beta \rightarrow \beta RF)). \end{aligned}$$

Here $F_{<a}$ uses R as its strict order. Define

$$\begin{aligned} \text{Closed}(Z) &\equiv Z0_\tau \wedge \forall \alpha^\rho (Z\alpha \rightarrow Z(\text{Suc } \alpha)) \\ &\quad \wedge \forall Y^{\rho \rightarrow t} ((\text{Rig } Y \wedge Y \subseteq Z) \rightarrow Z(\text{Lim } Y)), \\ \text{OT}_\tau &= \lambda \alpha^\rho. \forall Z^{\rho \rightarrow t} (\text{Closed}(Z) \rightarrow Z\alpha), \\ \alpha! &= \lambda RF. (\alpha RF \wedge \neg (\text{Suc } \alpha) RF). \end{aligned}$$

Thus the least-closed-collection notation is impredicative quantification, not an appeal to an external ordinal recursion. The definition of $H(A)$ in Lemma 3.4 now expands directly into these formulas.

For a carrier $Q : \sigma \rightarrow t$ and a non-strict strength order $D : \sigma \rightarrow \sigma \rightarrow t$ (so Ddc means d is stronger than c), put

$$\begin{aligned} \text{Down}(U) &\equiv U \subseteq Q \wedge \forall c, d \in Q ((Uc \wedge Ddc) \rightarrow Ud), \\ \text{Reg}(U) &\equiv \text{Down}(U) \wedge \forall c \in Q [Uc \leftrightarrow \forall d \in Q (Ddc \rightarrow \exists f \in Q (Dfd \wedge Uf))], \\ \text{RO}(Q, D) &= \lambda U^{\sigma \rightarrow t}. (\text{Rig } U \wedge \text{Reg}(U)). \end{aligned}$$

Its order is current inclusion restricted to this carrier. Cones are $\lambda d. (Qd \wedge Ddc)$. Compatibility is $\text{Comp}(c, d) \equiv \exists f \in Q (Dfc \wedge Dfd)$; a family $T \subseteq Q$ is an antichain iff $\forall c, d \in T (c \neq d \rightarrow \neg \text{Comp}(c, d))$. A family T is dense iff $T \subseteq Q \wedge \forall c \in Q \exists d \in T Ddc$.

For partial functions $c : A \times N \rightarrow 2$, eliminate the product notation by taking c of type $\alpha \rightarrow \nu \rightarrow \nu \rightarrow t$, where $A : \alpha \rightarrow t$ and $2 = \lambda i^\nu. (i = 0 \vee i = \text{Suc } 0)$. The partial-function condition is

$$\forall ani (cani \rightarrow Aa \wedge Nn \wedge 2i) \wedge \forall anij ((cani \wedge canj) \rightarrow i = j).$$

Its domain is $\lambda an.\exists i\ cani$, and extension is graph inclusion. Countability means that this two-place domain injects into N , using the tuple version of Inj. Finiteness means that it injects into some $N_{<m}$, $m \in N$. For completeness the arithmetic relation used here can be defined without recursive function symbols:

$$n \leq_N m \equiv Nn \wedge Nm \wedge \forall Z^{\nu \rightarrow t} [(Zn \wedge \forall k^\nu (Zk \rightarrow Z(\text{Suc } k))) \rightarrow Zm],$$

$$n <_N m \equiv (n \leq_N m \wedge n \neq m).$$

Countable closure itself is a formula: every $F : \nu \rightarrow \sigma \rightarrow t$ with $\text{Map}(F; N, Q)$ and

$$\forall n \in N \forall c, d \in Q ((Fnc \wedge F(\text{Suc } n)d) \rightarrow Ddc)$$

has a lower bound, that is, $\exists b \in Q \forall n \in N \forall c \in Q (Fnc \rightarrow Dbc)$. This is current closure, not necessary closure.

A.7 Real-number structures and CH

A real-number structure at carrier type η is a finite list (R, A, M, O, z, u) with types

$$R : \eta \rightarrow t, \quad A, M : \eta \rightarrow \eta \rightarrow \eta \rightarrow t, \quad O : \eta \rightarrow \eta \rightarrow t, \quad z, u : \eta.$$

The abbreviation $\text{Real}_\eta(R, A, M, O, z, u)$ is the conjunction of the following formulas. First $Rz \wedge Ru \wedge z \neq u$, and A, M are relational binary operations on R :

$$\forall xyv (Axyv \rightarrow Rx \wedge Ry \wedge Rv) \wedge \forall x, y \in R \exists! v^\eta Axyv,$$

with the same formula for M . In the remaining clauses, $x + y$ and xy are eliminable value notation for these relations, not terms asserted to exist in the original language. All the following algebraic identities are universally quantified over x, y, w in R ; z, u remain the fixed zero and unit parameters:

$$\begin{aligned} (x + y) + w &= x + (y + w), & x + y &= y + x, & x + z &= x, \\ (xy)w &= x(yw), & xy &= yx, & xu &= x, \\ x(y + w) &= xy + xw, \\ \forall x \in R \exists y \in R (x + y &= z), & \forall x \in R (x \neq z &\rightarrow \exists y \in R (xy = u)). \end{aligned}$$

Require also $\text{SO}(R, O)$ and

$$\begin{aligned} \forall x, y, w \in R (Oxy &\rightarrow O(x + w)(y + w)), \\ \forall x, y \in R ((Ozx \wedge Ozy) &\rightarrow Oz(xy)). \end{aligned}$$

Finally let $E_R = \lambda xy.(Rx \wedge Ry \wedge (x = y \vee Oxy))$ and require Dedekind completeness:

$$\forall X^{\eta \rightarrow t} [(X \subseteq R \wedge \exists x Xx \wedge \exists b \in R \forall x \in X E_R xb) \rightarrow \exists s^\eta \text{Sup}_{R, E_R}(X, s)].$$

These are the axioms of a Dedekind-complete ordered field; the Archimedean property follows, so need not be postulated separately.

To make the operation notation completely syntactic, recursively define value formulas for a finite algebraic expression s :

$$\begin{aligned} V_x(v) &\equiv v = x, & V_z(v) &\equiv v = z, & V_u(v) &\equiv v = u, \\ V_{s+r}(v) &\equiv \exists a^\eta b^\eta (V_s(a) \wedge V_r(b) \wedge Aabv), \\ V_{sr}(v) &\equiv \exists a^\eta b^\eta (V_s(a) \wedge V_r(b) \wedge Mabv). \end{aligned}$$

Replace $s = r$ by $\exists ab(V_s(a) \wedge V_r(b) \wedge a = b)$ and Osr by $\exists ab(V_s(a) \wedge V_r(b) \wedge Oab)$. This finite metalinguistic expansion eliminates every operation expression above; it introduces no quantification over syntax. The expressions $\exists \mathfrak{R} \text{Real}_\eta(\mathfrak{R})$ in the text abbreviate successive quantification over R, A, M, O, z, u at these types.

For a real-number structure with carrier R , define

$$\text{CH}(\mathfrak{R}) \equiv \forall X^{\eta \rightarrow t} [X \subseteq R \rightarrow (X \preccurlyeq N \vee R \preccurlyeq X)].$$

Because $X \subseteq R$ already gives $X \preccurlyeq R$, the second alternative is equivalent to equinumerosity by Cantor–Bernstein. This is the subset formulation used in both proofs, and does not presuppose that the continuum is well-orderable.

References

- [1] Richard Dedekind, *Stetigkeit und irrationale Zahlen*, Friedrich Vieweg und Sohn, Braunschweig, 1872. English translation, “Continuity and Irrational Numbers”, in *Essays on the Theory of Numbers*, translated by Wooster Woodruff Beman, Open Court, 1901. Cited for the classical cut construction; the asserted typed modal adaptation is not a result stated in that work.