Vagueness at every order

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Abstract

There are some properties, like *being bald*, for which it is vague where the boundary between the things that have it, and the things that do not, lies. A number of argument threaten to show that such properties can still be associated with determinate and knowable boundaries: not between the things that have it and those that don't, but between the things such that it is borderline at some order whether they have it, and the things for which it is not.

I argue that these arguments, if successful, turn on a contentious principle in the logic of determinacy: Brouwer's Principle, that every truth is determinately not determinately false. Other paradoxes which do not appear to turn on this principle often tacitly make assumptions about assertion, knowledge and higher order vagueness. In this paper I'll show how one can avoid sharp higher-order boundaries by rejecting these assumptions.

I used to be a child, but now I am not. It follows, given classical logic, that I stopped being a child at some point. Indeed, there was a first *nanosecond* at which I stopped: a nanosecond before which I was a child (or unborn) but at which I was not a child. Although such a nanosecond exists, it is a vague matter when it occurred: it happened during a period when it was borderline whether I was a child.

Different theories provide different accounts of what borderlineness *consists* in — perhaps it's ignorance, or semantic indecision, or something else. But those who accept classical logic agree about the above: logic ensures the existence of a boundary, and vagueness about where it lies is assuredly the diagnosis. I shall thus assume that anyone willing the theorize about vagueness understands the expression *it's borderline whether p*, or alternatively *it's vague whether p*. From this one may introduce, by definition, another expression, *it's determinate that p*, meaning simply: p and it's not borderline whether p. Determinacy, as

*A version of this paper has been circulating since 2009, at which time I received helpful feedback from many people, including Robbie Williams, Timothy Williamson, and especially Cian Dorr. I'd also like to thank audiences in Barcelona and Oxford around 2010 for helpful questions, and to the audience of the Aristotelian Society in 2020. The the newest draft of this paper is mostly the same in content but has been rewritten for stylistic reasons, apart from the model in section 7 which is new to this draft.

it is used in this paper, is a notion possessed by anyone with the concept of borderlineness, and its use does not require one to have particular views about what borderlineness consists in.[1]

The above reasoning extends. There were points in my childhood when I wasn't a borderline child: I was once a determinate child. But now I am not, so I must have stopped being a determinate child at some point: there was a last nanosecond of my determinate childhood. We should be no happier associating a precise length to my determinate childhood than my childhood so, as before, we should think that it is borderline at what nanosecond I ceased to be a determinate child. Thus among the determinate children, there are borderline determinate children and determinately determinate children.

This is the phenomenon of second-order vagueness. However, we may iterate the above considerations as many times as we like. Since I was once determinately a determinate child, and I stopped being so at some point, we should analogously posit borderline cases of determinate determinateness. Perhaps, then, there are some people who are determinately^n children for any amount of iterations, n, and some which are not. (Henceforth I shall write 'determinate^n' as shorthand for n successive 'determinately's.) Surely it is borderline where that boundary lies as well? In other words, there are some children such that it's neither borderline nor higher order borderline (borderline borderline, or borderline borderline borderline, and so on), whether they're children, and others such that it is either borderline or higher order borderline whether they are children, and it's borderline where the boundary between the two lies.

This is the phenomenon of higher-order vagueness, and this paper is concerned in particular with the possibility of vagueness in the last sort of case: between those who are borderline children at some order and those who are not. I'll present some arguments that purport to show that there is no vagueness here, show that they rest on a non-obvious principle in the logic of determinacy, and finally show how vagueness at all orders can be restored once the principle is rejected.

Section 1 argues that a satisfactory logic of determinacy ought to be compatible with vagueness at all orders. Section 2 expands on and formalizes a common argument to the effect that the distinction between the cases that are determinate at all orders and the rest must be precise. I argue that if this argument is to be successful, it must assume a principle in the logic of determinacy known as Brouwer's Priniciple: that whatever is true is determinately not a determinate false. Sections 3, 4 and 5 considers and rejects some other responses to this paradox. Sections 6 and 7 lay out the positive proposal and investigates various models of vagueness in which Brouwer's Principle is not valid. Section 7 presents a stronger paradox of higher-order vagueness which rests on a weakening of Brouwer's Principle I call Brouwer*, and ends by presenting natural models in which even the weaker principle is invalidated and in which there exists vagueness at all orders. Finally, in section 8 I relate the

[1]It is thus not, as sometimes alleged, a parochial concern of the supervaluationist, or others who use the word 'determinacy' in a more specific sense.

discussion to another well-known paradox of higher-order vagueness resting on a different contentious inference, known as Determinacy Introduction, and sketch a natural picture of vagueness which rejects the inference.

1 Does higher-order vagueness exist?

Some philosophers have sought to downplay the phenomenon of higher-order vagueness or deny its existence altogether.[2] By contrast, I will be using the ability to accommodate the existence of higher-order vagueness as a constraint on the logic of determinacy. Thus it is worth spending a few words on the importance of higher-order vagueness, and why it is urgent to have a theory that accommodates it.

Let's begin with the denial of second-order vagueness. According to that picture, the duration of your determinate childhood (unlike the duration of your childhood) is a precise length of time. Now typically, when you have a precise length of time, it is possible, given sufficient time and consideration, to figure out its length in some units such as nanoseconds. For instance, a year is a precise length of time, and although its length in nanoseconds requires some calculation, it is discoverable in principle. If the length of ones determinate childhood is precise, one might similarly expect to be able to find out its length in nanoseconds. Yet philosophers who deny second-order vagueness do not offer concrete hypotheses about how long they were determinate children, or about similar questions that have determinate answers by their lights. Anyone claiming that the length of their determinate childhood was exactly 378432178928476829 nanoseconds (say) would rightly be treated with incredulity.

There is evidently some obstacle to knowledge here. Of course, we also cannot know the precise length of our *childhood*, but there we have a ready explanation — vagueness — and many theories about how vagueness could prevent such knowledge to choose from. Someone who rejects second order vagueness owes us an alternative explanation. It would be quite astonishing if the explanation were different for our inability to know the length of our determinate childhood.

As with other vague properties, the property of being be a determinate child is also subject to a sorites paradox. We are just as inclined to accept the tolerance principle — that nanosecond difference could not make a difference to whether one is a determinate child, as we are the analogous tolerance principle for childhood. Soritesability is another hallmark of vagueness, and thus another reason to posit it here.

What I have said here applies equally to the higher orders of vagueness. If the length of my determinate determinate childhood was precise, we would expect to be able to determine that length. Since we cannot, it is natural to posit vagueness here: the alternatives are just as bewildering. This goes also for my

[2] See, for instance, Wright [29], Raffman [23], Burgess [7].

determinate determinate determinate childhood, my determinate determinate determinate determinate childhood, and so on.

The hallmarks of vagueness are unmistakable: their presence here is reason enough to posit vagueness. But what would happen if we pursued the other line of thought to its logical conclusion? Suppose that it is not vague when I stopped being a determinate child: there is some *sui generis* obstacle which prevents us from knowing. This other obstacle, whatever it may be, is every bit as puzzling as vagueness and calls out for philosophical theorizing. Let us then give it a name: *schmagueness.*

Once we can classify things as vague and schmague, we may introduce another operator, by analogy with the way we introduced determinacy from vagueness, meaning p and it is neither vague nor schmague whether p. Familiar soritical reasoning provides us with a last nanosecond during which it was neither vague nor schmague that I was a child. If the reasons for denying higher-order vagueness extend to higher-order schmagueness — as presumably they would given they are cognate notions — we should expect it to be neither vague nor schmague when this nanosecond occurred. We might thus posit a third notion, schm-schmagueness, to account for the obstacle to knowledge here. By iterating the above considerations in the evident manner, we must postulate an infinite hierarchy of different notions. (This picture, or something like it, is suggested by passages in Keefe [18].[3])

I think it should be clear, having followed this line of thought, that denying higher-order vagueness does not simplify matters particularly. Instead of higher-order vagueness you simply have a infinite hierarchy of concepts, and analogous puzzles involving them. In fact, there is a simple way to translate things I say in this paper that rest on the assumption that higher-order vagueness exists into claims which the hierarchical view described above would find acceptable. For once you have the concepts of vagueness, schmagueness, schm-schmagueness, etc., there is an umbrella concept of being one of these things: being vague, or schmague, or schm-schmague, etc. By reinterpreting the word 'vague' as it is used in this paper in terms of the umbrella concept, one must accept the existence of higher-order vagueness, and the limitative results to follow.In fact, the umbrella concept is arguably more centrally connected to the phenomena of vagueness than any particular level is — for instance, it straightforwardly corresponds with the distinctive obstacle to knowledge we encountered in the way that none of the levels of vagueness do. I would thus like to make a terminological suggestion: if you believe this phenomena associated with vagueness can be divided into levels in this way, going forwards, let us simply stipulate that the word 'vagueness' refers to the umbrella concept. (We'll return to the motivation behind this identification in the final section.)

[3]And also attributed there to Williamson [27] in a slightly different context. In her view, a new operator distinct from 'definitely' is be needed to capture the vagueness of 'definitely' (see p210).

2 A paradox of higher-order vagueness

Just as it is vague when your childhood ended, it is vague when your determinate childhood ended, as for your determinate determinate childhood, and so on. It is just as problematic, we have argued, to claim precision in the latter cases as in the former. However, some writers, tempting paradox, have suggested that precise cut-off points have to re-emerge at the boundary between between people who are determinately children at all orders — children who are not borderline children, not borderline borderline children, not borderline borderline children, and so on — and those who are not. The following passage by Mark Sainsbury is often cited in favour of this conclusion:

> Suppose we have a finished account of a [vague] predicate, associating it with some possibly infinite number of boundaries, and some possibly infinite number of sets. Given the aims of the description, we must be able to organize the sets in the following threefold way: one of them is the set supposedly corresponding to the things of which the predicate is absolutely definitely and unimpugnably true, the things to which the predicate's application is untainted by the shadow of vagueness; one of them is the set supposedly corresponding to the things of which the predicate is absolutely definitely and unimpugnably false, the things to which the predicate's non-application is untainted by the shadow of vagueness; the union of the remaining sets would supposedly correspond to one or another kind of borderline case. So the old problem re-emerges: no sharp cut-off to the shadow of vagueness is marked in our linguistic practice, so to attribute it to the predicate is to misdescribe it. [24]

Raffman [23], for example, describes this reasoning as 'decisive.' If it were correct, however, then it would indeed be paradoxical: there is no precise length associated with the period during which it was determinate at *all* orders that I was a child either. As with the finite orders, the property of being determinately a child at all orders bears all the hallmarks of vagueness: it is, for instance, susceptible to sorites reasoning, and we cannot put exact numbers on the length of time I have it.

Much turns on whether you accept classical logic or a non-classical logic. Sainsbury's conclusion that a given object either falls under a predicates realm of application, absolutely definitely and unimpugnably, or it fails to do so in some way, is equivalent to an instance of the principle of excluded middle. Perhaps there is some reason why this particular instance of excluded middle must be true, but Sainsbury's argument has done nothing to establish that. For the classical logician, however, Sainsbury's conclusion has no bite: what distinguishes a precise from a vague predicate is not whether it obeys the principle of excluded middle. We have not been given any reason to think that there can't be vagueness concerning whether something is absolutely definitely and unimpugnably true without a shadow of vagueness.

I think this dispenses with Sainsbury's argument as stated, but there is still something puzzling here. Let us grant, for the time being, this talk of claims being 'absolutely definitely and unimpugnably true, without a shadow of vagueness' (for short: 'true without a shadow of vagueness'). Then it seems that, on a sufficiently faithful regimentation of having a shadow of vagueness it should follow that a being vague whether or not you have a shadow of vagueness itself suffices for having a shadow of vagueness. Thus one should be able to assert:

> If it is vague whether p is true without a shadow of vagueness, then p in fact isn't true without a shadow of vagueness.

It would follow, then, that anything that is a borderline case of being true without a shadow of vagueness, isn't true without a shadow of vagueness. But this is not in itself an absurdity. One might feel unsettled by the more general thought that something could be a borderline case of Fness whilst failing to be F. But, given classical logic, this this sort of situation is actually quite pervasive. Let me explain why: Suppose it is borderline whether or not Harry is bald: it is borderline whether Harry is bald and it is borderline whether Harry is not bald. (The assumption of both these claims is strictly unnecessary, as either is the consequence of the other.[4]) An instance of the propositional tautology, $A \wedge B \rightarrow ((A \wedge \neg C) \vee (B \wedge \neg\neg C))$, yields that if it's borderline whether Harry is bald and borderline whether Harry is not bald, then either it is borderline whether Harry is bald and he isn't bald, or it is borderline whether Harry is not bald and Harry isn't not bald. In either case, we have found a case where something is borderline F without being F. What is more this situation must be utterly commonplace, as the preceding argument can be run with any example of a borderline proposition.

Let us press on. An absurdity can be reached from the assumption that it's vague whether p is true without a shadow of vagueness if, in addition to the previously indented conditional, we also had the following conditional:

> If it is vague whether p is true without a shadow of vagueness, then p is true without a shadow of vagueness.

Indeed, there is a certain principle in the logic of determinacy and borderlineness — Brouwer's principle — which would guarantee this conditional given a suitable regimentation of truth without a shadow of vagueness. If it is vague whether p is true without a shadow of vagueness, then it is not definitely false that p is true without a shadow of vagueness. However, it's definite that: if p is true without a shadow of vagueness then it is definitely so (if it were borderline then p would have a shadow of vagueness). So it is not definitely false that p is definitely true without a shadow of vagueness. However suppose we accepted the principle that anything that's not definitely not definitely true is in fact true (an equivalent of Brouwer's principle), allowing us to infer that p is in fact true without a shadow of vagueness.

[4]Which one can establish either by appealing to common sense, or fairly uncontentious assumptions about the logic of borderlineness.

It is clear that this argument would benefit from a little formalization. What do the adjectives 'absolutely', 'definitely' and 'unimpugnably' add in Sainsbury's argument? What does he mean by truth without a 'shadow of vagueness'. In what follows I shall interpret 'having shadow of vagueness' simply as vague at some order, and being true without a shadow of vagueness as being determinate at all orders. Let's introduce some notation.

- I shall say that it's determinate that p when p is true but not borderline. We will notate the claim that p is borderline ∇p, and notate the defined operator $p \wedge \neg\nabla p$ with Δp.
- We will write $\Delta^n p$ for the result of prefixing n of Δs to p, and we will pronounce it informally as 'p is determinate^n'.
- $\Delta^* p$ stands for the infinite conjunction $p \wedge \Delta p \wedge \Delta\Delta p \wedge ...$, and will pronounce it informally as 'p is determinate*' or 'p is determinate at all orders'.

A proposition is higher-order borderline if it is higher-order borderline at some finite order: borderline, borderline borderline, borderline borderline borderline, and so on. Finally, I shall use the word 'true' in the purely disquotational sense, in which one can, for instance, paraphrase the schema $\Delta A \to A$ as saying that whatever is determinate is true.[5]

Throughout this paper I assume classical logic. Since determinacy* is defined in terms of infinitary conjunction, I take this to include the obvious generalizations of the conjunction introduction and elimination rules to the infinitary case (that is, from $A_1, A_2, ...$ you may infer the conjunction $A_1 \wedge A_2 \wedge ...$ and from $A_1 \wedge A_2 \wedge ...$ you may infer A_1, A_2, et cetera.):

Classical Logic The axioms and rules of classical logic, including the obvious generalizations of these rules infinitary conjunction.

The reader may consult the supplementary document for the exact details.[6] To generate a paradox we will also need some assumptions governing the behaviour of Δ.

Necessitation If $\vdash A$ then $\vdash \Delta A$

Closure $\Delta(A \to B) \to (\Delta A \to \Delta B)$

[5] Without some disquotational device like this there is noeasy way to paraphrase many of the claims in this paper in ordinary English. It is possible that that some non-disquotational notion — e.g. supertruth — plays the other roles that truth is sometimes thought to play. But I find the use of the word 'true' for the disquotational notion both natural and convenient, and so will reserve technical words, like supertruth and determinate truth, for the non-disquotational notions.

[6] In the appendix I also consider a Hilbert axiomatization for classical infinite conjunction. In the Hilbert system the result only depends on classical tautologies, their generalizations to infinite conjunction, and modus ponens. It does not depend on which metainferences count as being part of classical logic — thus supervaluationists who reject metainferences like reasoning by cases must accept the following results too.

Brouwer's Principle $A \to \Delta\neg\Delta\neg A$

Necessitation is widely accepted as a closure condition on the logic of determinacy, thus anyone assigning the other three assumptions (Classical Logic, Closure and Brouwer's Principle) the status of logic can accept our uses of Necessitation. In fact, suppose we simply *define* the logic of determinacy to be those truths which are necessarily determinate at all orders. This seems like an eminently plausible thing to mean by 'the logic of determinacy' in this context. Then it easily demonstrated that the logic of determinacy is closed under the rule of necessitation.[7] Denying Necessitation then amounts to denying that one of the other three assumptions is necessarily determinate at all orders. We consider ways in which one might accept the truth of these three assumptions while denying they are determinate at all orders in section 5. The second principle, Closure, is also widely accepted, although we will briefly consider some dissenting viewpoints in section 4. The final principle, mentioned above, says that whatever is true is definitely not a definite falsehood. It is sometimes stated in the dual form $\neg\Delta\neg\Delta A \to A$ saying that whatever is not definitely not definitely true is true.

Notably absent from our assumptions is the principle that determinate truths are true, $\Delta A \to A$. This principle in fact falls out of our decision to treat borderlineness as primitive and determinacy as defined as a non-borderline truth, and so is not needed as an extra assumption. Had one made the opposite choice of primitives it would be presumably desirable to impose it as an additional axiom. However note that even if we made that alternative choice of primitives theorem below could still be proved without the assumption that Δ is factive.

With these sparse assumptions in place we can in fact *prove* analogues of the two indented conditionals above. For convenience, I shall state them in a contrapositive form that can be seen to be equivalent to those conditionals modulo definitions:

Theorem 1. *Given the above assumptions — **Classical Logic**, **Necessitation**, **Closure** and **Brouwer's Principle** — one can prove the following two conditionals:*

Positive Determinacy $\Delta^* A \to \Delta\Delta^* A$.

Negative Determinacy $\neg\Delta^* A \to \Delta\neg\Delta^* A$.

The contrapositive of Positive Determinacy corresponds to first indented conditional: if it is vague whether a proposition is determinate at all orders then it isn't in determinate at all orders. We will explain how it formally follows from our assumptions in the next section, and its derivation is in the supplementary

[7]Suppose A is necessarily determinate at all orders. Thus, necessarily, A, determinately A, determinately determinately A and so on. By dropping the first conjunct it follows that, necessarily, determinately A, determinately determinately A, and so on. This is just to say that determinately A is necessarily determinate at all orders. Later we shall introduce the more general notion of a *broad* necessity: a similar argument establishes that the broad necessities are also closed under Necessitation.

document. Negative Determinacy, which corresponds to the contrapositive of our second indented conditional, can be derived from Positive Determinacy using our assumptions as follows:

1. $\neg\Delta\Delta^* A \to \neg\Delta^* A$ contraposing Positive Determinacy
2. $\Delta(\neg\Delta\Delta^* A \to \neg\Delta^* A)$ by Necessitation.
3. $\Delta\neg\Delta\Delta^* A \to \Delta\neg\Delta^* A$ by from 2 by Closure
4. $\neg\Delta^* A \to \Delta\neg\Delta\Delta^* A$ an instance of Brouwer's Principle (simplifying the double negation in the consequent)
5. $\neg\Delta^* A \to \Delta\neg\Delta^* A$ from 4 and 3 by the transitivity of $\to$.

These two results deliver us with Sainsbury's conclusion: there is no vagueness concerning whether a proposition or sentence is determinate at all orders.

Corollary 2. *From **Classical Logic**, **Closure**, **Necessitation** and **Brouwer's Principle** we can prove $\Delta\Delta^* A \vee \Delta\neg\Delta^* A$, or equivalently, $\neg\nabla\Delta^* A$.*

The first form, for instance, follows from an instance of excluded middle, $\Delta^* A \vee \neg\Delta^* A$, and Positive and Negative Determinacy.

If there is never vagueness concerning what is determinate at all orders this means that we can, despite all our considerations to the contrary, associate a precise length of time to the period during which I was determinately a child at all orders. Since the assumptions of our theorem apart from Brouwer's Principle are mostly uncontentious, I take this to be a decisive reason to reject Brouwer's Principle. Still, there are other moves we might make in an attempt to Brouwe'rs Principle. I'll explore some of these in the following sections.

3 Independent arguments for Positive Determinacy

Several philosophers have maintained that Positive Determinacy is bad enough: that it alone rules out vagueness concerning what is determinate at all orders.[8] One upshot of Positive Determinacy is that if it is borderline whether someone is a determinate* child, then they are not a determinate* child (because if they were a determinate* child they'd have to be determinately so, given Positive Determinacy). Perhaps some of these philosophers are offended by the idea that something could be borderline F without being F. But we have already pointed out that this situation is not only consistent, but quite pervasive given our classical assumptions.

[8]See, for instance, Keefe ([18] chapter 8, p210), Field ([15], §11), Field (2008 §15.2-15.3). Williamson [27] p160 entertains this moral, but notes that without Brouwer's Principle certain sorts of higher-order vagueness are still possible.

See, for instance, Keefe [18], Field [16]. Williamson [27] entertains this moral, but notes that without Brouwer's Principle certain sorts of higher-order vagueness are still possible.

But in any case, Positive Determinacy is perfectly consistent with the claim that it is vague when ones determinate* childhood ended — we will consider models of this in section 7. More importantly, I think there is an independent and compelling argument in favour of Positive Determinacy, due to Williamson [27], from the plausible assumption that a conjunction of determinate truths is itself determinate:

$\Delta\wedge$ Distributivity $(\Delta A_1 \wedge \Delta A_2 \wedge ....) \to \Delta(A_1 \wedge A_2 \wedge ...)$

We may argue for Positive Determinacy thus: suppose A is determinate* so we have the conjunction $A \wedge \Delta A \wedge \Delta\Delta A \wedge ....$ Eliminating the first conjunct yields $\Delta A \wedge \Delta\Delta A \wedge \Delta\Delta\Delta A....$ So by $\Delta\wedge$ Distributivity we have $\Delta(A \wedge \Delta A \wedge \Delta\Delta A \wedge ...)$, which is just $\Delta\Delta^* A$ given our definitions.[9]

The argument presented in the last section for Positive Determinacy, based on Brouwer's Principle, is suspect for it is an argument for a stronger and more dubious conclusion: that there is no vagueness concerning what is determinate at all orders. But whatever we think of that argument, Williamson's argument makes a good independent case for accepting Positive Determinacy (thus forcing us to reject Negative Determinacy if we wish to avoid corollary 2).

However, not everyone is convinced. Hartry Field, for instance, takes Positive Determinacy (and various transfinite weakenings of it) to be a threat to his theory of higher-order indeterminacy, and provides a non-classical theory in which a conjunction of determinate truths needn't itself be determinate.[10] However, there is an even more basic argument for $\Delta\wedge$ Distributivity. The informal idea is just this: vagueness in a conjunction has to come from somewhere. If, *ex hypothesi*, it doesn't come from the conjuncts then it has to be coming from the operation of conjunction. But conjunction is surely precise. More generally we have the principle that conjunction is precise and precision is closed under application:

Conjunction is Precise The operation of conjunction (both finitary and infinitary) is precise.

Application Preserves Precision Applying a precise operation to precise arguments yields a precise result.[11]

Depending on ones philosophical sensibilities, we may interpret this principle as applying to bits of language, like names, predicates and sentences, or to more worldly things like individuals, properties and propositions. On the former

[9] This argument is presented as a conditional proof — one of the metainferences some supervaluationists reject — but it can be easily reformulated as an argument from propositional tautologies using modus ponens alone. Indeed, we may appeal to classical metainferences as we please so long as we stick to logical reasoning and avoid contentious Δ specific inferences like Determinacy Introduction (discussed in section 8). For then standard metatheorems, like the deduction theorem, guarantee that there is exists an alternative argument that uses only propositional tautologies and modus ponens.

[10] See note 41 in Field 2007 and the discussion on p270-271 in Field 2008, §17.4.

[11] This can be given a more precise statement in the setting of type theory. See, e.g., the formulation in Bacon [3] chapter 16 §1.

interpretation, for instance, the second principle correctly predicts that if you apply a precise predicate to a precise name you get a precise sentence as a result. It also follows that a conjunction of precise propositions is also precise, from which one can argue that a conjunction of determinate truths is itself determinate.[12]

It is also worth emphasizing that theorem 2 does not appeal to the principle that a conjunction of determinate truths is determinate. In fact, surprisingly, given the assumptions of that theorem — Classical Logic, Necessitation, Closure and Brouwer's Principle — one can actually derive $\Delta\wedge$ Distribution.

Theorem 3. *One can derive $\Delta\wedge$ Distribution from Classical Logic, Necessitation, Closure and Brouwer's Principle.*

The derivation can be found in the supplementary document.[13] Inspection of the derivation shows that very little distinctively classical reasoning is used — it would for instance, work equally well in a non-classical logic like Łukasiewicz logic or some of the logics Field has explored in recent years.

The moral is this. If we want to reject $\Delta\wedge$ Distribution and reinstate the higher-order vagueness Positive Determinacy rules out, it is possible as Field has demonstrated. But the bottom line remains the same: one still has to reject Brouwer's Principle.[14]

4 Closure

Closure is almost universally accepted. However there are a few voices of dissent: Susannah Bobzien [6] has suggested Closure might fail in relation to the paradoxes of higher-order vagueness, and Ofra Magidor has suggested the same but for unrelated reasons [19]. However the considerations in the previous section give us the tools to directly justify Closure. We have argued already that a conjunction of determinate truths is determinate — this applies just as forcefully to finite conjunctions as infinite. The converse of this principle just seems too

[12]There is a subtlety here, namely that a proposition or sentence might be determinate without being precise. For instance Patrick Stewart is determinately bald, but the proposition that he is bald is not a precise proposition — he might have had more hair and been borderline bald. Possible borderlineness is normally taken to be sufficient for a proposition to be vague. However, there are clearly connections between precision and determinacy: for instance a precise proposition cannot be borderline, it must be determinately true or determinately false. In Bacon [3] I argue from such connections that a determinate proposition is simply a proposition that's entailed by a precise truth. We may give a more rigorous argument for $\Delta\wedge$ Distributivity as follows: suppose that q_1, q_2, q_3 etc. are all determinate, with each being entailed by a precise truth p_1, p_2, p_3 etc. By our principle the conjunction of p_1, p_2, p_3 etc. is itself precise. It is also a conjunction of truths, so itself true, and entails the conjunction of q_1, q_2, q_3 etc. Thus the conjunction of q_1, q_2, q_3 etc is entailed by a precise truth, and so itself determinate.

[13]This derivation turns out to be formally similar to Lemmon's derivation of a structurally similar principle — the Barcan formula — from Brouwer's Principle. Prior [21] contains a derivation of the Barcan formula in $\mathsf{S5}$, and Prior [22] contains a derivation from Brouwer's Principle which he credits to Lemmon.

[14]Field's logic of determinacy in [16], for instance, does not contain Brouwer's Principle.

plausible to deny: you cannot have a determinate conjunction with a borderline conjunct. So if a conjunction is determinate, the conjuncts are too. Thus we have the biconditional:

$$(\Delta A \wedge \Delta B) \leftrightarrow \Delta(A \wedge B)$$

Suppose, also, that logical equivalents may be substituted *salve veritate*. Then we may reason as follows: the left-to-right direction of the biconditional yields the conditional $\Delta(A \rightarrow B) \wedge \Delta A \rightarrow \Delta((A \rightarrow B) \wedge A)$. $(A \rightarrow B) \wedge A$ is logically equivalent to $A \wedge B$; applying that substitution above yields $\Delta(A \rightarrow B) \wedge \Delta A \rightarrow \Delta(A \wedge B)$. Also $\Delta(A \wedge B) \rightarrow \Delta B$ using the right-to-left direction of the above biconditional. Chaining the last two conditional together yields $\Delta(A \rightarrow B) \wedge \Delta A \rightarrow \Delta B$, which is equivalent to Closure.

5 Nihilism*

It would be audacious to claim to have discovered that the length of ones determinate* childhood is both precise and exactly 378432178928476829 nanoseconds long. But there are less audacious ways to maintain that this length of time is precise, whilst providing a principled answer to its duration. One line of thinking has it that very little, perhaps even nothing, is determinate at all orders, so that I was never a determinate* child — the length of my determinate* childhood was exactly 0 nanoseconds. By analogy with the nihilist response to the sorites paradox, let us call this nihilism*.

Nihilist*s need not be nihilists — one can acknowledge the existence of children, and perhaps also determinate children, determinate determinate children, and so on for a while. But as the "determinately"s rack up, these extensions get smaller and smaller according to the nihilist* until they are empty. This is thus another way to respond to our paradox, for even if it is always a precise matter what is determinate at all orders, we can't prove the existence of a *precise* boundary without first assuming that there is a boundary at all. As with the nihilist response to the sorites, the nihilist* simply denies the existence of the relevant boundary.

I think the best versions of this response look slightly different depending on ones position on another contentious issue in the philosophy of vagueness: whether vagueness is primarily or fundamentally a property of propositions, or of linguistic entities like sentences or thoughts.[15] The dialectic in either case is similar, but different enough to warrant separate discussions. The former sort of person will typically try to explain linguistic vagueness in terms of propositional vagueness, whereas the latter will typically either explain propositional vagueness in terms of linguistic vagueness, or deny the existence of propositional vagueness altogether.

Let's start with the former. Friends of propositional vagueness are likely to think that metaphysical necessity is not the broadest sort of necessity. Some

[15] Most theorists are of the latter sort. Defenders of propositional vagueness include Schiffer [26] and Bacon [3].

propositions are necessary that are not determinate. Given the attractive idea that the vague supervenes on the precise, there will be claims concerning how cut-off points for vague properties depend on the underlying precise parameters that are metaphysically necessary, but not determinate.[16] It follows that metaphysical necessity is not broader than determinacy. (The converse inclusion clearly does not hold either.)

However, even once it is conceded that not all metaphysically necessary propositions are determinate, there are presumably some special propositions which are both metaphysically necessary and determinate. Indeed there will be special propositions that are necessary in every sense of 'necessary', such as the proposition that it's raining if it's raining. Let us give this distinction a name:

Broad Necessity A proposition p is *broadly necessary* iff every necessity operator applies to p.

By a necessity, here, I mean any operator that has the formal behaviour of a necessity, including operators like *determinately* and *always*.[17]

Finding uncontroversial examples of propositions that are broadly necessary is no easier than finding uncontroversial examples of metaphysical necessities: the answers will be highly sensitive to ones preferred metaphysics. Nevertheless, necessities, whatever they may be, are subject to some basic constraints, including, presumably the constraint that they are closed under logical consequence. From this we can substantiate our previous assertion that the proposition that it's raining is necessary, for every sort of necessity, and thus broadly necessary. (And more generally, propositions expressed by truths of logic are broadly necessary.[18])

Further plausible assumptions about meaning let us go further. Sometimes a single proposition can be expressed by two different sentences. If 'vixen' expresses the property of being a female fox, then 'vixens are female foxes' expresses the same proposition as the logical truth 'female foxes are female foxes'. More generally, it's reasonable to think that propositions expressed by 'analytic' or 'conceptual' truths are broadly necessary, including examples like the following:

1. Vixens are female foxes.
2. Scarlet is a shade of red.

[16] The easiest cases to think about are ones in which the underlying precise facts are not contingent: for instance, facts about how big or small numbers are relative to other numbers are necessary. A borderline small number thus will be either necessarily small or necessarily not small, as the case may be, but in either case it will not be determinate.

[17] See Bacon [3] chapter 11, Bacon [2] for a more precise formulation.

[18] Suppose $\Box$ is a necessity, then we might formalise the claim that $\Box$ is closed under first-order consequence by the schemata:

$\Box A$ when A is a theorem of first-order logic.

$\Box A \to \Box B$ when B follows from A in first-order logic

(This schematic way of treating the question is undoubtedly not entirely satisfactory. A general account of logical consequence as applied to propositions is presented in work with Jin Zeng currently in progress.)

3. Babies are children.

4. People with no hairs on their head are bald.

Notice, however, that once we have conceded this much, we have conceded enough to justify the major premise in a sorites argument for various problematic properties subject to theorem 2. For instance, since it's broadly necessary that scarlet is a shade of red, it must be determinate at all orders that scarlet is a shade of red.[19] But now we can consider various particular shades of colour further up the spectrum from *scarlet*, ordered so that adjacent shades differ only slightly. There must be a final shade which is determinately* red, and if the conclusion of theorem 2 is correct, there is paradoxically a determinate answer concerning which shade that is.

Linguistic theories of vagueness seem to fare better as vehicles for the nihilist* response. In [13], Cian Dorr argues that in evaluating whether a sentence is determinately true one must look at how that sentence is used by the relevant linguistic community in nearby worlds. As the iterations of the determinacy predicate rack up, one has to consider more and more distant possible worlds. Perhaps even worlds where logical words like 'not', 'or' and 'if ...then ...' are used in deviant ways, in which case even logically true sentences like 'it's raining if it's raining' might fail to express truths. The same can be said for the sentences 1-4.

But I think we should tread cautiously. Following Williamson [27], many have found it plausible that vague words could have had slightly different meanings than the ones they in fact have. But it less clear to me that they could have meanings as radically different as Dorr envisions. Consider, first, that there in fact exist different linguistic communities using words that sound and are spelled the same but have entirely different meanings. For instance, the German word 'gift' and the English word 'gift', which mean poison and gift respectively. In such cases, we shouldn't identify the words — in some cases they don't even belong to the same grammatical category! — we should rather just talk about different words that sound and are spelled the same. I think we should say the same about a possible world in which the people currently comprising the English speaking community were making noises and inscriptions exactly as the German speaking community presently do: the English word 'gift', for instance, is not being used differently in that world, but a different word — namely the German word 'gift' — is being used.[20]

Thus I am generally skeptical of the sorts of thought experiments Dorr invokes. I am in particular doubtful that completely precise words, like the logical

[19] More formally, determinacy at all orders behave formally like a necessity, so on any sufficiently liberal conception of what a necessity is, if every necessity applies to proposition 2, then 2 is determinate at all orders.

[20] Of course, if a word can be used with a slightly different meaning, but not an entirely different meaning, there is a puzzling question concerning how different these meanings can be — how much tolerance do our words have? These puzzles are similar in kind to the puzzles of material constitution — a wooden table can survive some amount of replacement of matter, but not complete replacement (see e.g. Chilsholm [9], Salmon [25]) — and we should expect a similar menu of possible responses.

constants, could be used with radically different meanings. Their present meanings are rather distinguished, and it would not be too far fetched to expect that any deviation from them would comprise a change of words. One might be similarly doubtful that words like 'vixen' could be prized apart from the meaning of 'female fox', 'scarlet' from 'red', et cetera, in ways that render 1-4 false, as would be impossible on a picture in which a words individuation conditions are holistically dependent on its relation to other words of the language.

6 Brouwer's Principle

I have argued that the correct logic of determinacy must accommodate vagueness at all orders. With the other assumptions of 2 vindicated, we are left to conclude that Brouwer's Principle is not be part of the logic of determinacy. However, it is one thing to reject a principle because paradox looms otherwise, and another to have a concrete picture of life without the principle. In this section and the next we shall develop some models of vagueness at all orders.

Few people have explicitly treated Brouwer's Principle principle in the context of the logic of determinacy, and fewer still have provided positive arguments for it. Let us begin by surveying them.

First, is the argument that Brouwer's Principle simply sounds initially plausible: every truth is determinately not a determinate falsehood. This is not an argument that has been put forward by anyone in particular, but it is an argument that deserves consideration anyway. It is worth emphasizing that generally we one should put little stock in such judgments. The notion of determinacy itself is highly theoretical. It is defined from another notion — borderlineness — and although I believe borderlineness is a notion we have pretheoretically, it is still one that requires a little philosophical instruction to latch on to. Judgments about the validity of principles involving the defined notion should be taken with a pinch of salt.

More substantively, there are many principles that sound as good as Brouwer's Principle which theory overturns pretty swiftly. Consider:

Every truth is determinately not a falsehood: $A \to \Delta\neg\neg A$

Every truth is a determinate truth: $A \to \Delta A$

Both may sound as compelling as the weaker principle that every truth is determinately not a determinate falsehood, but either quickly implies that there is no vagueness.[21] One possible diagnosis is that the word 'true', which I am using here purely in the disquotational sense, is easily conflated with determinate truth. In fact, if we make such a conflation in the statement of Brouwer's Principle we indeed get a valid principle:

Every determinate truth is determinately not a determinate falsehood.

[21] For example, two instances of the latter schema, $A \to \Delta A$ and $\neg A \to \Delta\neg A$, entail in the propositional calculus $\Delta A \vee \Delta\neg A$. This can be run for arbitrary A.

$\Delta A \to \Delta\neg\Delta\neg A$ can be derived from determinacy of the factivity of Δ, the determinacy of classical theorems and Closure.[22]

The only other consideration I know in favour of Brouwer's Principle is a purely theoretical one: there is a particularly simple and natural model of borderlineness, due to Williamson [27], according to which Brouwer's Principle is valid. Thus Williamson writes "Although these considerations in favour of B [Brouwer's Principle] are by no means decisive, we will provisionally include it as contributing to a particularly simple conception of the semantics, and call it into question again if it proves to have dubious consequences" [27] p130. I think theorem 2 is one such dubious consequence, so that we do indeed have reason to call it into question. But it is still instructive to see why Williamson's model predicts the truth of Brouwer's Principle, and what needs to be relaxed in order for it to no longer be a prediction.

Williamson's model has three ingredients:

- A set of indices, I
- A measure of distance between the indices, $d : I \times I \to \mathbb{R}$ (satisfying the axioms of a metric space).
- A margin of error, represented by a real number $\alpha \in \mathbb{R}$.

Indices should be thought of as entities that settle all questions. This not only includes contingent questions about precise matters — for instance, how old various people are in nanoseconds — but also as settling vague questions. For instance, an index must settle whether each person is a child or not, even in when it settles their age in nanoseconds to be within that period during which it is borderline whether they are a child. For the supervaluationist, indices may identified with ordered pairs of worlds and precisifications: the world coordinate settles questions concerning things such as who is what age in nanoseconds, and the precisification the vague questions, including who is and who is not a child given their ages in nanoseconds.[23] For the friend of propositional vagueness, a better interpretation of the indices is as broadly possible worlds: worlds that are possible in the broadest sense (and which will not always be metaphysically possible).[24] For concreteness I will adopt this interpretation going forwards, but the reader can translate back and forth as they like.

d is a measure of distance between indices: close pairs of indices will be assigned smaller numbers by d than distant pairs. Supervaluationists, and other linguistic theorists, sometimes gloss d as a measure how similar two indices are

[22] $\Delta((\Delta\neg A \to \neg A) \to A \to \neg\Delta\neg A)$ is obtained by prefixing a determinacy operator to the classical law of contraposition. Closure allows us to infer $\Delta(\Delta\neg A \to \neg A) \to \Delta(A \to \neg\Delta\neg A)$. The antecedent is an instance of the determinacy of factivity, so by modus ponens we get $\Delta(A \to \neg\Delta\neg A)$, which finally gives us $\Delta A \to \Delta\neg\Delta\neg A$ by closure again.

[23] Williamson is not a supervaluationist, but can adopt a similar reading of the indices as interpretations of a language that are for all we know correct.

[24] If Alex is a borderline child, then there will be two broadly possible worlds which agree about Alex's age in nanoseconds, but disagree about whether she is a child. Given the supervenience of childhood on age in nanoseconds, only one of these broadly possible worlds will be metaphysically possible.

with respect to the way the relevant language is used at them. However a more deflationary reading of the formalism is possible in which d simply measures the distance between the cutoff points between vague properties: for instance, if the only relevant property we are modeling is *childhood*, d could simply map a pair of indices to the difference between the cutoff points each index assigns to this property in nanoseconds. (The latter reading should also be more congenial to friends of propositional vagueness who deny any role to language use in explicating borderlineness.)

The informal idea behind Williamson's model of determinacy is this. I am a determinate child if I am well within the boundary of childhood. Thus if you were to vary the cutoff point for my childhood from its actual location, up or down by some margin, α nanoseconds, I would still count as being a child according to the modified cutoff point. If d represents the difference between the relevant cutoff points in nanoseconds, and i is the true index specifying the actual cutoff for childhood, then I should remain a child at all the indices j which place the cutoff point no further than α nanoseconds from i. Thus we say that:

ΔA is true at i iff A is true at every j such that $d(i,j) \leq \alpha$.

A slight variation of this clause replaces the $\leq$ with a $<$, in which case you must stipulate that $\alpha > 0$. Of course with the obvious clauses for negation, disjunction and conjunction (possibly infinitary) we have a model of the propositional language in which theorem 2 is stated.

We will deviate from Williamson by augmenting a model with a distinguished index $i \in I$ that represents the locations of the actual cutoff points. If there are indices in the model for each broadly possible cutoff point for each vague property, and since we know from classical logic that each property has a cutoff point, we know that at least one index gets them all correct. The existence of a distinguished index is thus no more contentious than classical logic is. (This point often not appreciated by supervaluationists, which sometimes leads to misplaced resistance to the idea that there is a distinguished precisification, even in this deflationary sense.) We finally say that a sentence is true in the model when it is true at the distinguished index.[25]

Brouwer's Principle is true at every index in these models. For suppose that A is true at i. Then for any j such that $d(i,j) \leq \alpha$, $\neg\Delta\neg A$ is true at j, since there is an A world within a distance of α from j (namely i). Thus $\Delta\neg\Delta\neg A$ is true at i, which means the conditional $A \to \Delta\neg\Delta\neg A$ is true at i. Alternatively, we may appeal to the fact that these models correspond to a Kripke frame (I, R) where the accessibility relation Rij holds when $d(i,j) \leq \alpha$. One of the axioms

[25] Williamson effectively equates truth in a model with being determinate at all orders. This strikes me as an extremely unnatural choice to make, and leads to some puzzling sounding consequences, such as the claim that second order vagueness leads to vagueness at all orders. A less contentious interpretation of this result would be simply that if it is determinate at all orders that a proposition is borderline borderline then for each n it is determinate at all orders that it is borderline^n. This result itself relies on Brouwer's Principle: see Mahtani [20] and Dorr [12].

of a metric is that the distance between i and j is the same as the distance between j and i ($d(i,j) = d(j,i)$) so this accessibility relation is symmetric: the exact condition under which Brouwer's Principle is valid in a frame.[26]

We explained the notion of determinate childhood in terms of the concept of being *well within* the boundary of childhood, which lead us to postulate that determinate truths must be true at all sufficiently near indices. But these glosses clearly invoke vague notions — being *well within*, *sufficiently near*, and so forth. And it is not clear that the vagueness gets eliminated when we move away from the informal glosses.[27] Yet in Williamson's model, these are all captured by a single parameter, α, which does not vary between the indices. In Williamson's model the notion of being *well within the boundary* gets represented as a completely precise notion.

Anna Mahtani suggests in [20] that we thus modify Williamson's semantics in a way that allows each index to be associated with its own conception of what counts as sufficiently near. Each index i will thus have an accessibility radius, $r(i)$, and a model is then:

- A set of indices, I
- A measure of distance between the indices, $d : I \times I \to \mathbb{R}$ (satisfying the axioms of a metric space).
- A margin of error, represented by a function from indices to real numbers $r : I \to \mathbb{R}$.

Mahtani does not impose any particular constraints on the function r, which leads to some pretty implausible models. Nearby points, for instance, can interpret *sufficiently near* in radically different ways, yet nearness is supposed to measure how close the interpretations are. Taking this into account would involve placing the constraints that prevent nearby points having radically different accessibility radii. We shall impose this by requiring that the difference between the accessibility radii at i and j should not exceed the difference between i and j:

(*) $|r(i) - r(j)| \leq d(i,j)$

A proposition A is then determinate at i if it is true at every index that is sufficiently near to i according to is criteria for being sufficiently near:

ΔA is true at i iff A is true at every j such that $d(i,j) \leq r(i)$.

As with the earlier model theory, there is also a strict version where $\leq$ is replaced with $<$ and we stipulate that $r(i) > 0$. Once each point has its own accessibility radius, the symmetry of accessibility is no longer guaranteed: the

[26] See, for instance, Hughes and Cresswell [11].

[27] For example, all kinds of factors contribute to how we classify people as bald – not just hair number, but distribution, colour and so on. Thus closeness of indices must be in part determined, in a weighted way, by how the indices match these various factors. How we weight these different factors is surely a vague matter.

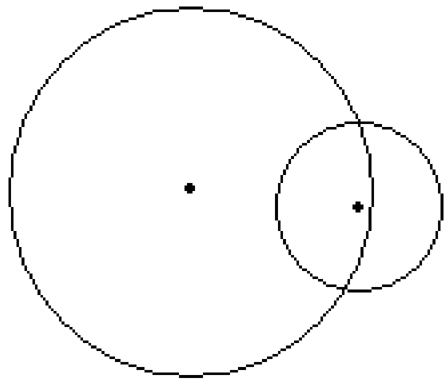

Figure 1: Mahtani's counterexample to Brouwer's Principle

distance between i and j may be less than $r(i)$ but not less than $r(j)$ (see figure 1). Mahtani's semantics is well motivated and not particularly complicated; this seems to me to undermine the theoretical argument for Brouwer's Principle. Perhaps it also raises the hopes of finding a realistic model of vagueness at all orders.

7 Revenge Paradoxes

Unfortunately we are not yet in the clear. In this section we will consider a further model theoretic argument, this time based on Mahtani's modification of Williamson's model, for our troublesome conclusion. These arguments will not be arguments in favour of Brouwer's Principle — we have just seen that it is invalidated in Mahtani's models — but some weaker principles that suffice to prove the same result: that there is no vagueness concerning what is determinate at all orders.

Let's begin with by stating a strengthening of Theorem 2. The assumption of Brouwer's Principle may be replaced, without loss, by any of the following weakenings.

Brouwer^n $A \to \Delta(B \to C_n(A, B))$

Brouwer* $\Delta(A \to \Delta A) \to \neg\Delta\neg A \to A$

where $C_1(A, B) := \neg\Delta\neg A$ and $C_{n+1} := \neg\Delta\neg(B \wedge C_n(A, B))$. Thus we have:

Theorem 4. *Positive and Negative Determinacy is derivable from Classical Logic, Closure, Necessitation, along with Brouwer^n for any n or Brouwer**

What do these weakenings of Brouwer's Principle actually mean? To obtain some insight, it is instructive to look at their frame conditions over ordinary

Kripke frames. Brouwer^n says that if an index i can see j, then it is possible to get back to i in n steps each of which i can see:

> If Rij then there are n points, $k_1, \ldots, k_n$ such that Rjk_n, Rk_nk_{n-1}, ..., Rk_2k_1, Rk_1i and for each $m = 1...n$, Rik_m.

Brouwer^1, which turns out to be just Brouwer's Principle given the definitions, thus corresponds to the symmetry of the accessibility relation. Brouwer* characterises the the property that if i sees j you can get back to i in *some* finite number of hops, each of which i can see:

> Rij then for *some* n, there are $k_1, \ldots, k_n$ such that Rjk_n, Rk_nk_{n-1}, ..., Rk_2k_1, Rk_1i and for each $m = 1...n$, Rik_m

Let us call a v-frame a tuple (I, d, r) defined as above, where I is a set of indices, d a metric, and r a radius function satisfying the condition (*) above. A v-frame validates a sentence when that sentence is true at every index in the frame. We have noted that v-frames do not validate Brouwer's Principle. In fact, it turns out that we can invalidate all of the above weakenings of Brouwer's principle. We can prove a completeness theorem that states that the logic of v-frames is exactly the modal logic KT, consisting of Necessitation, Closure and the Factivity of Δ, which is easily seen not to contain these weakenings of Brouwer's Principle.

Theorem 5. *The logic* KT *is sound and complete for* v*-frames.*

The proof is given in the supplementary document.

However the sorts of models that invalidate Brouwer^n and Brouwer* that are generated by the completeness theorem are quite contrived, and do not represent realistic models at all. To remedy this, let us try to develop some more realistic models. In our earlier toy model, we simply identified the indices with the cutoff it assigns to childhood in nanoseconds — thus I may be identified with the set of real numbers $\mathbb{R}$. The distance between indices $d(x, y) = |x - y|$ is just the difference between the cutoff points at x and y in nanoseconds. For now we will leave the radius function r unconstrained, apart from the general restriction that $|r(x) - r(y)| \leq d(x, y)$. For this discussion we will focus on the strict version of the model, i.e. in which determinacy corresponds to truth at all indices strictly within the radius.

We can extend these ideas to model more than one vague property at a time. For simplicity, we focus on properties that are determined by precise conditions that can be parametrized by real numbers. Suppose we wanted to model the properties of tallness and childhood together. The indices could then correspond to ordered pairs of real numbers, i.e. elements of the real plane $\mathbb{R}^2$, representing the cutoff for childhood and the cutoff for tallness in some units. Naïvely one might expect to be able to use the normal Euclidean notion of distance between pairs of points on the plane, as given by Pythogoras's theorem: $d((x, y), (x', y')) = \sqrt{(x - x')^2 + (y - y')^2}$.[28] However, this strikes me as the

[28] This metric is appealed to in, for instance, Dorr [12].

wrong choice. According to the Euclidean distance, a person is a determinate child *not* if the distance of their age from the boundary for childhood is below some threshold, but rather if the combination of the distance of their age from the boundary of childhood, and the distance of their height from the boundary of tallness doesn't exceed that threshold. (The combination in question is the squareroot of sum of the square of the distance of their age from the boundary of childhood and the square of the distance of their height from the boundary of tallness.) But intuitively these are independent properties: whether you are a determinate child depends only on your age and the boundary for childhood, not on your height. Rather a proposition is determinate when it is robustly true under perturbations of all the cutoff points individually. This suggests the correct metric (assuming this sort of model is adequate at all) is instead the so-called infinity metric $d((x,y),(x',y')) = max(|x-x'|,|y-y'|)$ given by the maximum of the two distances. In this metric the set of points within a given radius r of a point will look, if drawn using the incorrect Euclidean metric on $\mathbb{R}^2$, like a square around that point with edges of "length" $2r$.

The idea clearly extends to model n vague properties whose underlying precise conditions can be parameterized by numbers: the metric space of $\mathbb{R}^n$ with the infinity metric. The good news about the v-frames just described is that each of the problematic principles Brouwer^n are refutable. Indeed we can invalidate Brouwer^n in the simple case where the indices are $\mathbb{R}$, and the infinity and Euclidean metrics coincide. Let $\epsilon = \frac{1}{2^{n+1}}$. Stipulate that $r(0) = 1$, that $r(x) = r(-x) = \epsilon$ for $x \in \mathbb{R}\backslash[-1+\epsilon, 1-\epsilon]$ and $r(x) = 1-x$ for $x \in (0, 1-\epsilon]$ and $x-1$ for $x \in [\epsilon-1, 0)$. It is easy to check this satisfies condition (*) and is a v-frame. Now 0 can see 1, yet the shortest path back from 1 takes $n+1$ steps, thus B^n does not hold.

The counterexamples to Brouwer^n traded on the idea that for any n one can find a model like the one above based on an ϵ small enough to ensure that the longest path from 1 to 0 is longer than n. There is, however, no single model which refutes all the Brouwer^n simultaneously. Indeed it is not hard to show that v-frames based on $\mathbb{R}^n$ based on the strict account of determinacy has the backtrack property: if x sees y, then there is a finite path $z_0, \ldots, z_n$ such that y sees z_0, z_0 sees z_1, $\ldots$, z_n sees x and x sees each z_i. Thus it follows that Brouwer* holds in these frames (see figure 2). Intuitively, the closer a point is to x, the closer in diameter it's accessibility range must be to x's thus one always can find a path leading back to x. This result in fact doesn't depend on whether we use the Euclidean or infinity metric (replacing the circles with squares in figure 2 won't change the bottom line).

If this argument is correct it puts us back to square one. For if Brouwer* is valid then, given our other assumptions, theorem 4 tells us that the length of my determinate* childhood is completely precise — a sort of revenge paradox, even once one has rejected Brouwer's Principle.

The route to this conclusion was undoubtedly serpentine: we should careful to avoid some common pitfalls. This argument for Brouwer*, like the preceding argument for Brouwer's Principle, is theoretical in nature: it depends on a very particular sort of model theory. Model theory is an excellent tool for

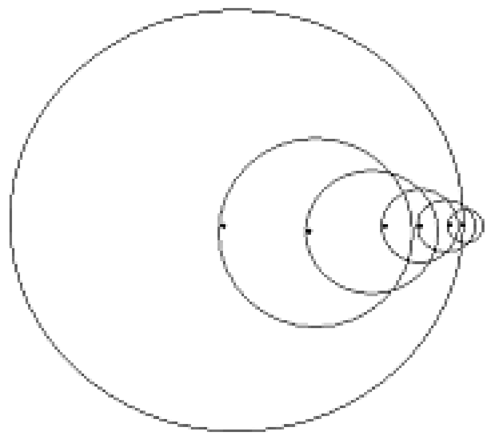

Figure 2: The validity of Brouwer*: x sees a far out point on the edge, yet there is a finite path back to x.

testing theories for consistency and for exploring consequences of particular assumptions. But it is less useful as persuasive device: the predictions of a model are only as good as the model itself. Simplifying assumptions may turn out to be not as innocuous as they initially seem, as we have seen already with Williamson's original model theory, which included the assumption that the accessibility radii are constant. We also made several unforced decisions in the models considered above. For instance, the results above only hold in the strict version of the model theory, according to which determinacy consists in truth over the points within but not including the relevant radius. One can have vagueness concerning what is determinate at all orders in the non-strict version.[29] More importantly, the model theory employs a single radius which can be used for all the different possible parameters: age, height, et cetera. This made certain metrics less natural, like the standard Euclidean metric, and forced us towards the infinity metric. However, a different approach would be to associate several radii to each world, one for each vague predicate: how far from the boundary of childhood or tallness you need to be in order to be a determinate child or determinately tall may vary from index to index — in this augmented version they needn't vary in tandem.

I will not go into all the possible ways one might improve the models above. Instead I will describe a simple and relatively intuitive model of childhood, which predicts that there is vagueness about the boundary of determinate* childhood. Let us suppose that the actual cutoff point for childhood is at N units of time, and that $N-K, ..., N, ..., N+K$ lists the range of broadly possible cutoff points

[29]For instance, let $I=[0,1]$ with the standard metric and $r(x)=1-x$. Any sentence that is true at 1 but not 0 is such that at 0 it is borderline whether it is determinate at all orders.

for childhood in that unit of time. Someone who is $N-K$ units old is thus not determinately a child at all orders, although they may be a child, or even a determinate child. The broadly possible worlds in the model will consist of ordered pairs (n,k) of natural numbers where n is between $N-K$ and $N+K$ and $0 \leq k \leq N-K$. According to the world (n,k), n is the cutoff point for childhood, and $n-k, ..., n+k$ is the range of broadly possible cutoff points for childhood according to that world. Thus, crucially, what is broadly possible can vary from world to world. It remains to specify the accessibility relation. At a world (n,k) we know that everyone $\leq n-k$ nanoseconds is a determinate* child. We also know, given Positive Determinacy (which we justified in section 3 by Williamson's argument) that if people $\leq n-k$ units of time are determinately* children, then they are determinately determinate* child, and thus they are also determinate* children at any point (n,k) sees. *The length of ones determinate* childhood can get longer from one accessible point to another, but it cannot get shorter.* Or, equivalently, the range of broadly possible cutoff points can shrink from one accessible world to another, but it cannot expand. This means, for instance, that (n,k) cannot see $(n-1,k)$ for according to that world $n-1-k$ is a broadly possible cutoff point. Lastly, we should also expect accessible worlds to differ slightly over the cutoff point for childhood, in line with the Williamsonian idea that to be a determinate child is to remain a child under small shifts in the location of the cutoff for childhood. I shall assume that we have chosen our units of time so that the relevant relevant threshold is just 1 unit of time. To summarize:

- $I = \{(n,k) \mid k \geq 0 \text{ and } N-K \leq n-k \leq n \leq n+k \leq N+K\}$
- $R(n,k)(n',k')$ iff $(n,k) = (n',k')$ or $|n-n'| \leq 1$ and $k' = k-1$
- The true world: (N,K)

Brouwer* is not valid in this frame: k only gets smaller as you move along the accessibility relation, so once you've accessed a new world you can never find your way back. We can use this model to find borderline cases of determinacy at all orders (and thus counterexamples to the conclusion of theorem 4). Since each step from (n,k) decreases k, and k cannot go below 0, $n-k$ is the smallest cutoff point for childhood that can be accessed by repeatedly following the accessibility relation, and $n+k$ is the largest. E.g. to get $n-k$ as the cutoff for childhood (the first coordinate) from (n,k) one can follow the the route $(n,k) \to (n-1,k-1) \to ... \to (n-k,0)$. Thus the cutoff for childhood at (n,k) is n and the cutoff for determinate* childhood at (n,k) is $n-k$.

Consider the proposition that I was a determinate* child at $N-K+1$ nanoseconds. Since the cutoff for determinate* childhood at the world (n,k) is $n-k$, this proposition is the set of points where $n-k \geq N-K+1$: $p = \{(n,k) \mid n-k \geq N-K+1\}$. This proposition is thus false at the true world (N,K). But it's true at $(N,K-1)$ since $N-(K-1) \geq N-K$. So if I am exactly $N-K+1$ nanoseconds old, then is is borderline whether I'm a determinate* child: an example of borderline determinacy at all orders.

We described the above model as a Kripke model. However it can, in fact, be redescribed as a sort of generalization of Mahtani's model mentioned above, in which each index comes with a pair of accessibility radii, which may vary from world to world: in this case, one for childhood and another for determinate* childhood.[30]

8 Other Paradoxes of Higher-order Vagueness

Having defended the possibility of vagueness in what is determinate at all orders, let me conclude by relating my discussion to another class of paradoxes of higher-order vagueness which purport to show that there must be precision even among the finite iterations of determinacy. These paradoxes start with claims that formalize the thought, defended in section 1, that, for each number of iterations n, it is vague where the boundary of my determinate^n childhood lies. Following Graff Fara's [14] influential presentation of this paradox, I will call these the Gap Principles, and may be formalized as follows:[31]

Gap Principle If I was determinately a determinate^n child at k nanoseconds, then I wasn't determinately not a determinate^n child an $k+1$ nanoseconds.

$$\Delta\Delta^n C_k \to \neg\Delta\neg\Delta^n C_{k+1}$$

The argument then purports to reduce these assumptions to absurdity. Apart from assumptions already defended above, namely Classical Logic and Closure, the only other principle used in the derivation is the principle of Determinacy Introduction stated below:

Determinacy Introduction From A one may infer ΔA.

$$A \models \Delta A$$

Before we proceed, it is worth emphasizing how this principle is different from the rule of necessitation. This is best illustrated by looking at a case where our new rule is uncontroversially *invalid*: the logic of metaphysical necessity. In that context the rule of necessitation allows one to prefix theorems of ones logic with a $\Box$ — a reasonable rule given the plausible assumption that the relevant theorems express metaphysical necessities. By contrast, the analogue of Determinacy Introduction is clearly invalid, as allows one to infer the necessity of ones assumptions: from the widely accepted assumption that snow is white, one could infer that it was necessary that snow is white, a falsehood.

It is rare to find clear articulation of why Determinacy Introduction is a good form of inference. It is noteworthy, however, that many supervaluationists have described model theories in which Determinacy Introduction is valid. At

[30]Specifically, we use $d((n,k),(n',k')) = max(|n-n'|, \sum_{i=k}^{k'} \frac{1}{2}^i)$ as the metric and $r(n,k) = (1, \frac{1}{2}^k)$ as the radius function.

[31]An earlier version appears in Wright [28]. See also Zardini [30] for another helpful perspective.

least, in some sense of the world 'valid': there is actually some controversy over the meaning of the word 'valid', in this context and in general, but I think it is mostly beside the point and has made some of the subsequent discussion hard to untangle. Rather than get embroiled this controversy, I think it would be much more productive to simply look at what sorts of things this inference preserves.

In this context, there are three salient statuses that an inference might have. It might preserve truth (i.e. disquotational truth), it might preserve determinate truth, and it might preserve determinate truth at all orders. An inference is valid in the first (second, third) sense iff necessarily, whenever the premises are true (determinate, determinate at all orders), so is the conclusion.[32]

I am inclined to think that it is a purely terminological matter which of these three statuses should be associated with the honorific title of 'valid argument'. But at any rate, it is only in the last sense that the rule of determinacy introduction is straightforwardly valid. If the inference from A to ΔA preserved disquotational truth then all conditionals of the form $A \to \Delta A$ would be true, from which it immediately follows that there is no borderlineness whatsoever.[33] The assertion that determinacy introduction preserves determinate truth amounts to the claim that whatever is determinate is determinately determinate, $\Delta A \to \Delta\Delta A$: this is the controversial $\mathsf{S4}$ axiom for determinacy which rules out most forms of second-order vagueness. By contrast, the rule of determinacy introduction does preserve determinacy at all orders: if A is determinate, determinately determinate, determinately determinately determinate, and so on, then (eliminating the first conjunct) A is determinately determinate, determinately determinately determinate and so on. This is just to say that if A is determinate at all orders, then determinately A is determinate at all orders.

What does it mean when we derive a contradiction from an assumption, A, using any of these three notions of inference? In the first case it means that it is impossible that A is true, in the second case it means that it is impossible that A is determinate, and in the third it means it is impossible that it be determinate at all orders. Here are some illustrative examples: $A \wedge \neg A$ is inconsistent in all three senses, and in particular cannot be true. By contrast, $A \wedge \neg\Delta A$ is inconsistent in the second and third senses, but not the first: it thus can't be determinate or determinate at all orders, but it could be true. Finally $A \wedge \neg\Delta^* A$ is inconsistent only in the third sense: it is consistent that it be both true and determinate, but it cannot be determinate at all orders.

Since the Gap Principles — which encode the claim that the length of my determinate^n childhood is vague — lead to absurdity according to the third sense of validity it follows that they cannot be determinate at all orders. But does that mean the length of my determinate^n childhood is *not* vague? No: it is, in fact, perfectly consistent that the length of my determinate childhood is vague, that the length of my determinate determinate childhood is vague,

[32] Most theorists employ either the first or third notion, although Cobreros [10] argues that supervaluationists should adopt the second.

[33] $((A \to \Delta A) \wedge (\neg A \to \Delta\neg A)) \to (\Delta A \vee \Delta\neg A)$ is an instance of the classical tautology $((A \to B) \wedge (\neg A \to C)) \to (B \vee C)$, and the antecedent is given by conjoining two instances of the schema.

and so on. Indeed, the model we described in section 7 corroborates all these assertions: at the true world (N, K) all of the Gap Principles are true.

The Gap Principles presumably have a more secure status than simply being true. We showed, in section 2, that even truths can be borderline, yet presumably the Gap Principles are not borderline. We seem, for instance, to be in a position to assert (as I did earlier) that the length of my determinate^n childhood is not precise. Yet we shouldn't assert borderline truths. Similarly, we seem to be in a position to know that the length of my determinate^n childhood is not precise, and borderlineness here would preclude this too. Thus we have good reason to believe that the Gap Principles are not only true but determinate. It is easy to verify that the Gap Principles are also determinate at (N, K) in our model.

Of course, I have just asserted the the Gap Principles are determinate, and in order for *this* assertion be proper it should also be determinate, which amounts to the claim that the Gap Principles are determinately determinate. And now I have just asserted that they are determinately determinate, and in order for *that* assertion to be proper they should be determinately determinately determinate! But can I continue this performance forever, each time justifying the determinacy of the previous assertion by appealing to the assumption that it was made properly? I think not: just because I have made an assertion properly, doesn't mean that I am in position to properly assert that my assertion was made properly. This situation should be familiar to those who accept the knowledge norm of assertion: one might know p without knowing that you know p, and thus be in a position to assert p, without being in a position to assert that you are in a position to assert p. But even without accepting the knowledge norm we can see why this reasoning is illegitimate. *Being in a position to assert* is as vague as any relation. For instance, I'm in a position to assert that I was a child a 0 nanoseconds, and not in a position to assert that I was a child at 100000000 nanoseconds, so there must be a n such that I am in a position to assert that I was a child a n nanoseconds, but not in a position to assert the same about $n+1$. Of course, this is a case where, although I am in a position to assert that I was a child at n nanoseconds, it is borderline whether I am in a position to assert this. Thus just from the principle that one shouldn't assert borderline propositions, we may infer that I cannot assert that I am in a position to assert, even though I *am* in a position to assert.

It is just as well that asserting the Gap Principles does not commit you to their determinacy at all orders, since the upshot of Fara's argument, we have shown, is that they are *not* determinate at all orders. This is seen quite clearly in our model of the last section: most worlds in that model see other worlds apart from themselves. The exception are the worlds of the form $(n, 0)$: since the second coordinate (the k) must always decrease as you move to a new accessible world, and cannot go below 0, these worlds only see themselves. According to worlds that only see themselves, everything is precise: ΔA and A become equivalent, and $\Delta A \vee \Delta \neg A$ is true at these worlds for all sentences A. Since there is no vagueness at these worlds, there is no vagueness concerning the length of my determinate^n childhood — the Gap Principles are false. One might find it odd that our model contains worlds in which there is no vagueness, however, on the interpretation of the indices I've been employing, they are merely *broadly* possible worlds. They are neither metaphysically possible, or possible in the sense of the determinacy operator, they are actually quite distant worlds: you cannot get to one from (N, K) in less than K steps. They are thus things that are not determinately^K false.

I think this dispenses with Fara's argument. Nevertheless, since I have claimed that we may not assert propositions unless they are determinate, and I have also claimed that we may assert some propositions that fail to be determinate at all orders (like the Gap Principles), one might wonder what principled account of the relation between borderlineness and assertion could predict this. The view I defend in Bacon [1] and Bacon [3] (chapter 7) is simply this: borderlineness, and borderlineness alone, generates that distinctive vagueness related barrier to knowledge (and consequently assertion). If it is borderline whether it's borderline that Harry is bald, and Harry is in fact not borderline bald — he's determinately bald — then you are, barring other more mundane reasons for ignorance, in a position to know that Harry is bald. Rather, it will be borderline whether you are in a position to know that Harry is bald, and also borderline whether you are in a position to assert this. This picture goes along naturally with the view, which we suggested adopting in section 1, that vagueness is just *identified* with the source of that distinctive kind of ignorance, whatever that might be (and more generally is identified with whatever is behind whatever else you take to be the hallmark of vagueness). If it is vague whether something is vague, then it is simply vague whether it gives rise to that distinctive sort of ignorance.

This does not appear to be the dominant position in the philosophy of vagueness. According to orthodoxy, borderline borderlineness is also a distinctive source of ignorance. This goes equally for third and higher-order borderlineness: they all do the same work of preventing knowledge, and doing whatever else first-order vagueness does. According to this view, if you are in a position to assert something, then it cannot be borderline, or higher-order borderline: it must be determinate at all orders. This position is often associated with supervaluationism, but it is pretty widespread . It is explicit in much of Hartry Field's work (see e.g. [16]) and in Kit Fine's, both of whom endorse a non-classical logic. Fine, for instance, writes: 'In asserting some propositions P1,P2, ..., one is committed to more than their actual content, one is also committed to their being definitely the case, definitely definitely the case, definitely definitely definitely the case, and so on' (Fine [17] p114).

But this makes it puzzling what the role of borderlineness operator really is: borderlineness on this picture is not what causes that distinctive kind of ignorance, and is associated with the other phenomena characteristic of vagueness. Second-order borderlineness, third-order borderlineness, and so on, also cause these things. Instead it is the umbrella concept — being borderline or higher-order borderline — that plays the important theoretical role and is responsible for the phenomena that has vexed philosophers of vagueness. Borderlineness itself is just an idle wheel: something that must be infinitely iterated in order to generate the theoretically useful notion — the one that governs proper assertion, knowledge, uncertainty and so on. If this is your view, what is the use of borderliness in the first place? Wouldn't it be simpler to relabel the umbrella concept 'borderlineness' and theorize in the way I have suggested? Doing so, I submit, leads to a much more satisfactory account of vagueness.

9 Appendix

9.1 The Distribution of Determinacy over Conjunction

In this section we prove the following theorem.

Theorem 6. *One can derive $\Delta\wedge$ Distribution from Classical Logic, Necessitation, Closure and Brouwer's Principle.*

$\Delta\wedge$ Distributivity $\bigwedge_{i<\omega} \Delta A_i \to \Delta \bigwedge_{i<\omega} A_i$.

The assumptions, recall, are

Closure $\Delta(A \to B) \to (\Delta A \to \Delta B)$

Brouwer's Principle $A \to \Delta\Diamond A$

Necessitation if $\vdash A$ then $\vdash \Delta A$

Classical Logic is taken to include the following principles of infinitary conjunction:

C1. $\bigwedge_{i<\omega} A_i \to A_n$ for each $n < \omega$.

C2. $\bigwedge_{i<\omega}(A_i \to B_i) \to (\bigwedge_{i<\omega} A_i \to \bigwedge_{i<\omega} B_i)$.

C3. If $\vdash A_i$ for each $i < \omega$, $\vdash \bigwedge_{i<\omega} A$.

For convenience I shall introduce an operator $\Diamond p := \neg\Delta\neg p$.

Although the distributivity of Δ over infinite conjunctions is independent of K (we establish this below), the distributivity of $\Diamond$ over infinite conjunctions, perhaps surprisingly, is not independent in this way and can be show given just some relatively uncontroversial principles governing infinite conjunction.

Lemma 7. *$\Diamond \bigwedge_{i<\omega} p_i \to \bigwedge_{i<\omega} \Diamond p_i$ is derivable from C1-C3, Closure and Necessitation.*

Proof. First note that $\Diamond \bigwedge_{i<\omega} p_i \to \Diamond p_j$ for each j, by C1 and the background modal logic of K. Then by C3 and then C2 we can infer $\Diamond \bigwedge_{i<\omega} p_i \to \bigwedge_{i<\omega} \Diamond p_i$. □

We may now prove theorem 3 as follows:

Proof. Brouwer's Principle directly gives us:

$\bigwedge_{i<\omega} \Delta p_i \to \Delta\Diamond \bigwedge_{i<\omega} \Delta p_i$

We may also infer from our lemma that

$\Delta\Diamond \bigwedge_{i<\omega} \Delta p_i \to \Delta \bigwedge_{i<\omega} \Diamond\Delta p_i$

by applying necessitation and the K principle. Finally we have

$\Delta \bigwedge_{i<\omega} \Diamond\Delta p_i \to \Delta \bigwedge_{i<\omega} p_i$

because we have $\Diamond\Delta p_i \to p_i$ for each i by B. So by C3 and then C2 we get $\bigwedge_{i<\omega} \Diamond\Delta p_i \to \bigwedge_{i<\omega} p_i$, and by necessitation and K that gives (12).

But the three equations above give Distributivity. □

This establishes $\Delta\wedge$ Distributivity given our assumptions. It should be noted that this argument does not appeal to any characteristically classical principles. Indeed this argument can be carried out provided one has the following rules of inference as primitive or derived:

$\to$1. $A, A \to B \vdash B$

$\to$2. $A \to B, B \to C \vdash A \to C$

$\to$3. $A \to B \vdash \neg B \to \neg A$

And we take both versions of Brouwer's Principle as axioms: $A \to \Delta\Diamond A$ and $\Diamond\Delta A \to A$. (Without the second version the most we could prove without double negation elimination would be things of the form $\Diamond\Delta\neg A \to \neg A$.) Given Closure and Necessitation as background assumptions, this suggests that in order to reject $\Delta\wedge$ Distributivity in a non-classical framework one must also reject Brouwer's Principle.

Note also that our derivation made *essential* use of Brouwer's Principle. Without it we could not prove $\Delta\wedge$ Distributivity as the following independence result shows.

Proposition 8. *$\Delta\wedge$ distributivity is independent of Closure, Necessitation, Factivity and C1-C3.*

Proof. Construct a Montague-Scott frame as follows (see Chellas [8]): Let $\mathcal{W} := \mathbb{N}$ and for each world $w \in \mathcal{W}$ let the necessary propositions at w, $N(w)$, be the cofinite subsets of $\mathcal{W}$ (if we are trying to model Factivity as well we let $N(w) := \{X \cup \{w\} \mid X \text{ is cofinite}\}$.) Then $\langle \mathcal{W}, N\rangle$ satisfies:

1. $N(w)$ is nonempty.
2. $X \in N(w)$ and $X \subseteq Y$ then $Y \in N(w)$.
3. If $X, Y \in N(w)$ then $X \cap Y \in N(w)$
4. (For Factivity) If $X \in N(w)$ then $w \in X$.

Thus our frame models our assumptions including C1-C3. However it does not model $\Delta\wedge$ Distributivity, as can be seen by letting $[\![p_i]\!] := \{n \in \mathbb{N} \mid n > i\}$ (for KT: $\{n \in \mathbb{N} \mid n > i\} \cup \{0\}$, allowing $\Delta\wedge$ Distributivity to fail at 0.) On the other hand any Kripke frame (reflexive Kripke frame) will validate K (KT) along with $\Delta\wedge$ Distributivity. □

9.2 Positive and Negative Determinacy

In this section we establish the following theorem.

Theorem 9. *Given the above assumptions — **Classical Logic**, **Necessitation**, **Closure** and **Brouwer's Principle** — one can prove the following two conditionals:*

Positive Determinacy $\Delta^* A \to \Delta\Delta^* A$.

Negative Determinacy $\neg\Delta^* A \to \Delta\neg\Delta^* A$.

We first show how to derive Positive Determinacy

Proposition 10. *Positive Determinacy follows in Classical Logic from Classical Logic, Brouwer's Principle, Closure and Necessitation).*

This is essentially the argument given in Williams [27].

Proof. Firstly we note that

$$\bigwedge_{n<\omega} \Delta^n A \to \bigwedge_{n<\omega} \Delta^{n+1} A$$

is provable from C1-C3: it corresponds to eliminating the first conjunct. We have $\bigwedge_{n<\omega} \Delta^n A \to \Delta^i A$ for each $0 < i < \omega$ by C1. Thus $\bigwedge_{i<\omega}(\bigwedge_{n<\omega} \Delta^n A \to \Delta^{i+1} A)$ by C3. So finally $\bigwedge_{n<\omega} \Delta^n A \to \bigwedge_{i<\omega} \Delta^{i+1} A$ by C2.) By theorem 3 we can derive $\Delta\wedge$ Distributivity from the stated assumptions. By $\Delta\wedge$ Distributivity we can immediately infer

$$\vdash \bigwedge_{n<\omega} \Delta^n A \to \Delta \bigwedge_{n<\omega} \Delta^n A$$

which is, given definitions:

Positive Determinacy $\Delta^* A \to \Delta\Delta^* A$.

□

Proposition 11. *Negative Determinacy follows in Classical Logic from Classical Logic, Brouwer's Principle, Closure and Necessitation.*

Proof.

1. $\neg\Delta\Delta^* A \to \neg\Delta^* A$ contraposing Positive Determinacy.
2. $\Delta(\neg\Delta\Delta^* A \to \neg\Delta^* A)$ by Necessitation
3. $\Delta\neg\Delta\Delta^* A \to \Delta\neg\Delta^* A$ from 2 by Closure
4. $\neg\Delta^* A \to \Delta\neg\Delta\neg\neg\Delta^* A$ an instance of Brouwer's Principle
5. $\neg\Delta^* A \to \Delta\neg\Delta\Delta^* A$ substituting logical equivalents.
6. $\neg\Delta^* A \to \Delta\neg\Delta^* A$ by 3 and 5 and transitivity of the conditional.

Note that step 5 relied on the substitution of logical equivalents which is a derived rule given our assumptions. Thus we have

Negative Determinacy $\neg\Delta^* A \to \Delta\neg\Delta^* A$.

□

9.3 The logic of v-frames

Here we demonstrate some relevant facts about the logic of vagueness in v-frames. Recall that:

Definition 1. *A v-frame is a triple $\langle W, d(\cdot,\cdot), r(\cdot)\rangle$ where $\langle W, d\rangle$ is a metric space, and $r : W \to \mathbb{R}^+$ obeys the following:*

(A) $\forall w, v \in W, |r(w) - r(v)| \leq d(w, v)$

A formula of propositional modal logic is valid on a v-frame $\langle W, d, r\rangle$ iff it is valid on the Kripke frame $\langle W, R\rangle$ where Rxy iff $d(x, y) \leq r(x)$.

Dorr [12] shows, translating into the terminology of v-frames, that B is not valid over the v-frame $\langle (1, 2), |x - y|, \frac{x}{3}\rangle$ although the weaker principles $p \to \Delta\neg\Delta\Delta\neg p$ and B^2 are valid in this frame. It is possible, however, to construct v-frames in which $p \to \Delta\neg\Delta^n\neg p$ is valid for no $n \in \mathbb{N}$. For example, let $W := \{0, 1\}$, $d(x, y) = |x - y|$, $r(0) = 1$ and $r(1) = \frac{1}{2}$.

What is the logic of v-frames? Clearly every v-frame generates a corresponding reflexive Kripke frame, so the logic of v-frames contains KT. One might have hoped that every reflexive Kripke frame could be generated from a v-frame this way ensuring a logic of exactly KT. This reduces to the question of whether every reflexive Kripke frame can given the structure of a metric space in such a way that there is a a closed ball around each node that contains all and only those nodes it can see. Unfortunately this does not hold:

Proposition 12. *Suppose $\mathcal{F}$ is a Kripke frame based on a v-frame. If $\mathcal{F}$ contains a cycle, it contains a 2-cycle.*

Proof. To see this suppose that $\langle a_0, \ldots, a_n\rangle$ is a cycle in $\mathcal{F} = \langle W, R\rangle$ where $n > 2$. For convenience let $a_i = a_j$ where $j = i \bmod (n + 1)$ for $i > n$. Now suppose that that $\neg Ra_{i+1}a_i$ for every i. Since for each i Ra_ia_{i+1} we know that $d(a_i, a_{i+1}) \leq r(a_i)$ in the corresponding v-frame. We also know that $r(a_i) < d(a_{i-1}, a_i)$ since $\neg Ra_ia_{i-1}$. Thus for each i, $d(a_i, a_{i+1}) \leq r(a_i) < d(a_{i-1}, a_i)$, so $r(a_n) < d(a_{n-1}, a_n) \leq r(a_{n-1}) < \ldots \leq r(a_{-1}) = r(a_n)$, i.e. $r(a_n) < r(a_n)$ which is a contradiction. So for some i, Ra_ia_{i+1} and $Ra_{i+1}Ra_i$. □

v-frames thus have more structure than reflexive frames. However, it turns out this does not make a difference to the logic:

Theorem 13 (Completeness)**.** *A set Σ is valid on every v-frame iff it's members are theorem's of* KT.

Proof. Suppose that Σ is a KT-consistent set of formulae. Then Σ is satisfiable on the canonical frame $\mathcal{F}$. $\mathcal{F}$ may contain cycles without 2-cycles, so we cannot yet infer that Σ is satisfiable on some v-frame. However we may construct a frame from $\mathcal{F}$, with all the cycles ironed out by a standard technique called 'unraveling' (see e.g. Hughes and Cresswell [11] chapter or Blackburn et al. [5]). The result is equivalent to a v-frame.

Let a_0 be a maximal KT-consistent set containing Σ. We may assume that a_0 is a root of $\mathcal{F}$ (if it isn't take the generated subframe around a_0 and work with that instead.) Define $\mathcal{F}^+ := \langle W^+, R^+\rangle$ as follows

- $W^+ := \{s \mid s \text{ a path in } \mathcal{F} \text{ such that } s_0 = a_0\}$
- $R^+ := \{\langle s, t\rangle \mid |t| = |s| + 1 \text{ and } s_i = t_i \text{ for } i \leq |s| \text{ or } s = t\}$

Fact $f(a_0, \ldots, a_n) = a_n$ is a bounded morphism from $\mathcal{F}^+$ to $\mathcal{F}$, and any set of sentences satisfiable on $\mathcal{F}$ is satisfiable on $\mathcal{F}^+$ (see for example Blackburn et al. [5]).

It follows that Σ is satisfiable on $\mathcal{F}^+$. Now we construct our v-frame as follows:

- We begin by defining distance between adjacent points. If R^+st then $e(s,t) = e(t,s) = \frac{1}{2^{|s|}}$. Always fix $e(s,s) = 0$
- $d(s,t) := inf\{\sum_{i=0}^{n} e(p_i, p_{i+1}) \mid p_0 = s, p_n = t, p \text{ a path in the symmetric closure of } \mathcal{F}^+\}$
- $r(s) := \frac{1}{2^{|s|}}$

It is now easy to check that $\langle W^+, d, f\rangle$ is a v-frame and that R^+st iff $d(s,t) \leq r(s)$. $\square$

Cian Dorr has pointed out to me that the constraint (A) on v-frames does not play much of a role in the proof of our completeness theorem. This allows us to prove a slightly more general result:

Definition 2. *a* **difference measure** *is a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that:*

- *g is continuous in both arguments.*
- $g(x,x) = 0$
- $g(x,y) = g(y,x)$ *(this constraint is not needed in what follows, but seems independently desirable.)*

For a given difference measure, g, a g-frame is a triple $\langle W, d(\cdot,\cdot), r(\cdot)\rangle$ where $\langle W, d\rangle$ is a metric space, and $r : W \to \mathbb{R}$ such that:

(A′) $\forall w, v \in W, g(r(w), r(v)) \leq d(w,v)$

Corollary 14. *For any difference measure g, the logic of g-frames is* KT.

Proof. Note that for any positive a there is a $b < a$ such that $g(a,b) \leq a$ since $g(a,a) = 0$ and g is continuous in both arguments. For any a pick a unique such b, a_g (choice.)

Now modify the construction in Theorem 4.1 as follows.

- Fix $e(s,s) = 0$ for every s.
- Let $e(\langle a_0\rangle), t) = e(t, \langle a_0\rangle) := 1$ for $t \neq \langle a_0\rangle$ such that $R^+\langle a_0\rangle, t$.
- Suppose that $e(s,t) = e(t,s) = a$ has already been defined for R^+st, and suppose that R^+tu $t \neq u$. Define $e(t,u) = e(u,t) = a_g$.
- $d(s,t) := inf\{\sum_{i=0}^{n} e(p_i, p_{i+1}) \mid p_0 = s, p_n = t, p \text{ a path in the symmetric closure of } \mathcal{F}^+\}$

- $r(s) := sup\{e(s,t) \mid R^+st\}$.

□

The interest in this generalization is that one might think that it becomes much harder for two points to differ on the interpretation of 'determinately' the closer together they are. Perhaps it is not the difference between $r(w)$ and $r(v)$ that must be less than $d(w,v)$ but the difference between their ratios, or some other such g.

9.4 Weakenings of Brouwer's Principle

In this section we prove some facts about the weakening of Brouwer's Principle we called Brouwer* in section 7.

Brouwer* $\Delta(p \to \Delta p) \to (\neg\Delta\neg p \to p)$

In what follows we shall refer to this principle as simply B^*.

Proposition 15. *B^* is valid in the class of v-frames just described. In the presence of KT, B^* defines the backtrack principle:*

> *Whenever Rxy there exists $z_1, \ldots, z_n$ such that (a) $z_1 = y$, $z_n = x$ and Rz_iz_{i+1} for $1 \le i < n$ and (b) Rxz_i for $1 \le i \le n$.*

Proof. We shall show that $(\Delta(p \to \Delta p) \wedge \neg\Delta\neg p) \to p$ defines the requisite property. Suppose the reflexive frame $\mathcal{F} = \langle W, R\rangle$ has the backtrack property. Now suppose $x \Vdash (\Delta(p \to \Delta p) \wedge \neg\Delta\neg p)$. The second conjunct ensures that there is a y such that Rxy and $y \Vdash p$. Since $\mathcal{F}$ has the backtrack property there is a finite path back from y to x, $z_1, \ldots, z_n$, which x can see. Since $x \Vdash \Delta(p \to \Delta p)$ each $z_i \Vdash p \to \Delta p$. Since $z_1 = y$ and $y \Vdash p$, $y \Vdash \Delta p$ - by induction we can see that $z_i \Vdash p$ for each i which means $z_n = x \Vdash p$ as required.

For the other direction suppose, for contradiction, that $\mathcal{F} \models (\Delta(p \to \Delta p) \wedge \neg\Delta\neg p) \to p$ but $\mathcal{F}$ lacks the backtrack property. This means that for some x and y, Rxy but there is no path back from y to x which x can see. Define the following valuation on $\mathcal{F}$: $w \Vdash p$ iff there are $z_1, \ldots z_n$ such that (1) $z_1 = y$, Rz_nw and Rz_iz_{i+1} for $1 \le i < n$ and (2) Rxz_i for $1 \le i \le n$. Certainly if x had this property then $z_1, \ldots, z_n, x$ would be a path back to x which x can see, so $x \nVdash p$. However $x \Vdash \Delta(p \to \Delta p)$ since if Rxw and $w \Vdash p$ then there is a path from y to w satisfying (1) and (2): $z_1, \ldots z_n$. Furthermore, for any world that w sees, w', $z_1, \ldots, z_n, w$ will be a path from y to w' satisfying (1) and (2), since Rxw.

□

In section 7 we argued that B^* is valid in v-frames based on metric spaces of the form $\mathbb{R}^n$ where the radius function function determined a open, rather than a closed ball around each $a \in \mathbb{R}^n$. In these frames whenever x can see y, there is a path back from y to x. Is KTB^* the modal logic of these v-frames? We start with a negative result: KTB^* is not sound and *strongly* complete with respect to *any* class of frames.

Proposition 16. *There is no class of frames, $\mathcal{C}$, such that a set is KTB^* consistent iff it's satisfiable on a frame in $\mathcal{C}$.*

Proof. To show this we shall show there is a KTB^*-consistent set of sentences which is unsatisfiable on every frame validating KTB^*.

Let $\Sigma := \{p, \neg\Delta\neg q\} \cup \{\Delta(q \to \Delta^n \neg p) \mid n \in \omega\}$. If Σ were KTB^*-inconsistent some finite subset would be KTB^*-inconsistent (since proofs are finite.) We shall show that for every $m \in \omega$, $\Sigma_m := \{p, \neg\Delta\neg q\} \cup \{\Delta(q \to \Delta^n \neg p) \mid n \in m\}$ is KTB^*-consistent. Σ_m has a KTB^*-model: $\langle m+1, R\rangle$ where Rxy iff $x = 0$ or $x > 0$ and $|x - y| \leq 1$. 0 can see m and there is a finite m length path back from m to 0 that 0 can see but no shorter path. Let q be true only at m and p only at 0.

However, if $\mathcal{F}$ validates KTB^* then $\mathcal{F}$ has the backtrack property so at no point of $\mathcal{F}$ is every member of Σ true: if $x \Vdash \neg\Delta\neg q$ then x sees some $y \Vdash q$. By the backtrack property there is a path $z_1, \ldots, z_n$ back to x which x can see, so $\Delta(q \to \Delta^{n+1}\neg p)$ cannot be true at x if $x \Vdash p$.

□

However there is a positive result, namely that KTB^* is sound and complete over the class of reflexive frames with the backtrack property. For this result I refer the reader to Benton [4], who shows that KTB^* has the finite model property.

Theorem 17. *If ϕ is KTB^*-consistent then it is satisfiable on a finite reflexive frame with the backtrack property.*

I have not been able to settle the question of whether KTB^* the logic of v-frames based $\mathbb{R}^n$ with open accessibility radii.

References

[1] Andrew Bacon. Can the classical logician avoid the revenge paradoxes? *Philosophical Review*, 124(3):299–352, 2015.

[2] Andrew Bacon. The broadest necessity. *Journal of Philosophical Logic*, 47(5):733–783, 2018.

[3] Andrew Bacon. *Vagueness and Thought.* Oxford University Press, 2018.

[4] R.A. Benton. A Simple Incomplete Extension of T which is the Union of Two Complete Modal Logics with F.M.P. *Journal of Philosophical Logic*, 31(6):527–541, 2002.

[5] P. Blackburn, M. De Rijke, and Y. Venema. *Modal logic.* Cambridge University Press, 2002.

[6] Susanne Bobzien. If it's clear, then it's clear that it's clear, or is it? higher-order vagueness and the s4 axiom. In B. Morison K. Ierodiakonou, editor, *Episteme, etc.: Essays in honour of Jonathan Barnes.* OUP UK, 2012.

[7] J.A. Burgess. The sorites paradox and higher-order vagueness. *Synthese*, 85(3):417–474, 1990.

[8] B.F. Chellas. *Modal logic: an introduction*. Cambridge Univ Pr, 1980.

[9] Roderick M. Chisholm. Identity through possible worlds: Some questions. *Noûs*, 1(1):1–8, 1967.

[10] Pablo Cobreros. Supervaluationism and logical consequence: A third way. *Studia Logica*, 90(3):291–312, 2008.

[11] M. J. Cresswell and G. E. Hughes. *A New Introduction to Modal Logic*. Routledge, 1996.

[12] C. Dorr. How Vagueness Could Cut Out At Any Order. *Unpublished*, 2010.

[13] C. Dorr. Iterating Definiteness. *Cuts and Clouds*, 1(9):550–586, 2010.

[14] D.G. Fara. Gap principles, penumbral consequence, and infinitely higher-order vagueness. *Liars and Heaps: New Essays on Paradox*, pages 195–222, 2003.

[15] Hartry Field. Solving the paradoxes, escaping revenge. In J. C. Beall, editor, *Revenge of the Liar: New Essays on the Paradox*. Oxford University Press, 2007.

[16] H.H. Field. *Saving truth from paradox*. Oxford University Press, USA, 2008.

[17] K. Fine. The Impossibility of Vagueness. *Philosophical Perspectives*, 22(1):111–136, 2008.

[18] R. Keefe. *Theories of vagueness*. Cambridge University Press, 2000.

[19] Ofra Magidor. Epistemicism, distribution, and the argument from vagueness. *Noûs*, 52(1):144–170, 2018.

[20] A. Mahtani. Can Vagueness Cut Out at Any Order? *Australasian Journal of Philosophy*, 86(3):499–508, 2008.

[21] A. N. Prior. Modality and quantification in s. *Journal of Symbolic Logic*, 21(1):60–62, 1956.

[22] Arthur Prior. *Past, Present and Future*. Clarendon Press, 1967.

[23] D. Raffman. Demoting higher-order vagueness. *Cuts and Clouds*, pages 509–522, 2010.

[24] R.M. Sainsbury. Concepts without boundaries. *Vagueness: a reader*, pages 251–264, 1996.

[25] Nathan Salmon. The logic of what might have been. *Philosophical Review*, 98(1):3–34, 1989.

[26] Stephen Schiffer. Vague properties. In Richard Dietz and Sebastiano Moruzzi, editors, *Cuts and Clouds: Vagueness, its Nature, and its Logic*, pages 109–130. Oxford University Press, 2010.

[27] T. Williamson. *Vagueness*. Routledge, 1994.

[28] C. Wright. Is higher order vagueness coherent? *Analysis*, 52(3):129, 1992.

[29] C. Wright. The illusion of higher-order vagueness. *Cuts and Clouds*, pages 523–549, 2010.

[30] E. Zardini. Squeezing and stretching: How vagueness can outrun borderlineness. In *Proceedings of the Aristotelian Society (Hardback)*, volume 106, pages 421–428. John Wiley & Sons, 2006.