Logical Necessity

Andrew Bacon
Kit Fine

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Some philosophers have posited a distinctively logical notion of necessity – *logical necessity* – under which logical truths alone are necessary.[1] This posit leads to the following two questions:

1. Logical truths are usual taken to be sentences, necessary truths to be propositions. Is there then a propositional notion of logical necessity corresponding to the linguistic notion of logical truth and, if there is, then how is the correspondence to be made out?

2. Granted that there is a propositional notion of logical necessity, then what is its logic?

These questions will be our principal concern in what follows. However, there are some other questions one might wish to consider. These include the question of whether logical necessity is the strongest form of necessity, the question of whether other notions of necessity can be explained in terms of logical necessity, and the question of whether the notion of logical necessity can itself be explained in other terms. We shall briefly touch on these other questions, and we refer the reader to a fuller treatment of them in the references.

1 Logical Truth and Logical Necessity

We suppose given a primitive propositional operator, $\Box$. This is to be understood as logical necessity, so that '$\Box \ldots$' for a given sentence '$\ldots$' may be read: it is a logically necessity that .... Our intention is that the understanding of logical necessity should be appropriately related to our understanding of logical truth. However, logical truth, as it is typically spelled out, is a predicate of sentences – attaching to a name of a sentence to form a sentence, whereas logical necessity is typically a propositional operator – attaching to a sentence (not a name of a sentence) to form a sentence. There is therefore no straightforward way of explicitly defining the one in terms of the other. What we may do instead is to impose some constraints on how the two notions are to be related.[2]

As a representative example of such constraints, we might demand that the sentence:

It is logically necessary that Hesperus is identical to Hesperus

[1] The notion of logical necessity appears throughout the history of philosophy; see for instance the collection Rini et al. (2016).

[2] This problem was brought into prominence by Quine; see Quine (1953)

be true since the sentence 'Hesperus is identical to Hesperus' is a logical truth; and we might also demand that the sentence:

> It is logically necessary that Daniel Bernoulli was the son of Johan Bernoulli

be false since the sentence 'Daniel Bernoulli was the son of Johan Bernoulli' is not a logical truth. In general, we may stipulate that 'it is logically necessary that' is to be interpreted in such a way as to make the following true for any sentence A of a given language $\mathcal{L}$:

Bridge Principle 'It is logically necessary that A' is true if and only if 'A' is a logical truth.

It should be recognized that this principle is not only schematic in the sentence A but also in the language $\mathcal{L}$ and in the notion of logical truth. Thus the import of the principle is different depending upon what we take the language $\mathcal{L}$ to be and what we take logical truth to be. In what follows we will consider variations along these two parameters, but it will be important for us always to allow the language $\mathcal{L}$ itself to contain the operator $\Box$ for logical necessity. Logical truths may themselves involve the notion of logical necessity.

The above principle raises a number of interesting philosophical questions. One is whether we should think of the logical possibilities as real or substantive possibilities as to how things might be. It is logically possible that Daniel Bernoulli had different parents. But do we want to say that there is a real possibility that he had different parents even though it is not metaphysically possible that he had different parents.[3] There are different ways one might go on this question but, fortunately, our use of the Bridge Principle as a constraint will not require us to take a stand.

There is also a question as to whether we might want to beef up the Bridge Principle and whether we might, in particular, want to treat the statements on the right of the biconditional as providing some kind of reduction of the statements on the left of the biconditional. This is the most natural way of construing the "provability interpretation" of modal logic (as in the work of Solovay (1981) or in the formulation of the modal paradoxes in Montague (1963)). The Bridge Principle is treated, in effect, as the means by which a modal statement may, in any context in which it occurs, be replaced or

[3]There is a purely verbal issue we should set aside: in some of the earlier literature, 'logical necessity' was sometimes treated as just another term for metaphysical necessity, and as a result many claims were counted as 'logically necessary' which are not logically necessary in our sense.

"grounded" by a corresponding predicative statement. But again, our approach is neutral on this question, as it is on the question of whether one might wish to go in the reverse direction and reduce logical truth to logical necessity.

A more substantive question for us concerns the application of the Bridge Principle when the given language contains the identity predicate or the idioms of quantification. The application of the Principle to quantified statements is not straightforward since it applies most directly to closed not to open sentences and so would appear to call for a substitutional interpretation of the quantifiers. We deal with this question in section 4 but let us here focus on the case of identity. Given that Hesperus is identical to Phosphorus, it follows that:

> it is logically necessary that Hesperus is identical to Hesperus if and only if it is logically necessary that Hesperus is identical to Phosphorus.

But under the conventional view in which 'Hesperus is identical to Hesperus' is a logical truth while 'Hesperus is identical to Phosphorus' is not, the left-hand-side of the above biconditional will be true and the right-hand-side false.

There are a number of ways out of this difficulty. Fine has proposed an account of logical truth under which 'Hesperus is Phosphorus' is a logical truth by requiring that it is the referents of the names, rather than the names themselves, which should figure in the logical form.[4]

Another solution is to suppose that we are working within a language which never contains two constants for the same thing. Within such a language, there will then be no true non-trivial identities, such as 'Hesperus is identical to Phosphorus', and so the previous problem will not arise. We might take this proposed solution one step further and suppose that we are working within what Russell calls a logically perfect language:

> In a logically perfect language the words in a proposition would correspond one by one with the components of the corresponding fact, with the exception of such words as "or", "not", "if", "then", which have a different function. In a logically perfect language, there will be one word and no more for every simple object, and everything that is not simple will be expressed bya combination of words, by a combination derived, of course, from the words for the simple things that enter in, one word for each simple component.' Russell (1940), p.25.

[4]See Fine (1989) and Fine (1990).

In such a language there will be no two simple names for the same thing and, also, no simple expression for something complex — a simple predicate 'vixen', for example, for the complex property of being a female fox. Thus just as the problem over the logical necessity of Hesperus being identical to Phosphorus will no longer arise, nor will the problem over the logical necessity of the property of being a vixen being the same as the property of being a female fox. Moreover, such a constraint on the language is very natural given that our aim, in setting up the Bridge Principle, is to project the logical features of language down onto the logical features of the world. We might also weaken the constraints on a logically perfect language by broadening the notion of logical truth. We might, for example, follow Frege in the Grundlagen and take a sentence to be an analytic truth if S logically follows from some suitable set of identities or definitions.[5] We could then say that it is an analytic necessity that vixens are female foxes given that 'vixens are female foxes' follows, by definition, from 'to be a vixen is to be a female fox'. But, of course, how far we can go in this direction is very much an open question.[6]

2 Logical Truth

There are several accounts of logical truth—model theoretic, proof-theoretic, substitutional, and the like[7]—which might be plugged into the Bridge Principle to obtain a notion of logical necessity. Common to almost all accounts of logical truth are the satisfaction of two constraints: (i) that logical truths be true[8]; and (ii) that they be closed under substitution.

The first of these constraints is relatively straightforward.[9] The second calls

[5] Frege (1884).

[6] For an older discussion of this question, see Pap (1958).

[7] See [GRIFFITHS AND PASEAU] this volume.

[8] Since we have taken $\mathcal{L}$ to be an interpreted language, we have a notion of truth *simpliciter* for sentences of $\mathcal{L}$. Going forward we will assume, without comment, that truth behaves classically with respect to the truth functional connectives. That is, a conjunction is true iff both conjuncts are, the negation of a sentence is true iff the sentence is not true, and so on.

[9] Though not altogether straightforward, since it conflicts with certain model-theoretic accounts of logical truth. Consider a language with a primitive logical cardinality quantifier, 'there are numerous x', for which it is true that there are numerous sets but not true that any set has numerous members. Combine this with a model theoretic account of logical truth where being logically true amounts to being true across a class of models whose domains of quantification are given by a set. Then the sentence 'there are numerously many things' is true whilst being a logical falsehood. One might, of course, take this to reveal a problem for the proposed account of logical truth rather than for the proposed constraint.

for more explanation. It presupposes a distinction between the logical and the non-logical constants. Here we count $\Box$ itself as a logical constant, along with the other familiar logical constants such as $\wedge$ and $\forall$.[10] A *substitution* is then an operation that maps expressions to expressions by uniformly replacing each non-logical constant in an expression with another expression of the same logical type; and a set of expressions will be *closed under substitution* if the result of applying a substitution to any expression in the set will also lie in the set. Such a rule is naturally taken to encode the idea that logic is topic-neutral – that whereas the special sciences may make distinctive claims concerning a specific subject-matter, the laws of logic will hold regardless of the subject-matter. So if we take the subject-matter of a claim to be given by its non-logical constants, then the logical truth of such a claim should not turn on which particular non-logical constants are used in the formulation of the claim.

Our investigation will be restricted to theories that meet a third constraint, that the set of logical truths should contain the standard axioms of classical logic (appropriate to the given language) and be closed under the standard rules of classical logic. We do not wish to be dogmatic on this matter, but the treatment of non-classical logic lies outside the scope of the present paper. We call a set of sentences meeting these three conditions a *sound logic*. We will sometimes also consider a condition stronger than classicality, in which the logical truths must constitute a normal modal logic. We summarize the relevant distinctions in the following definition.

Definition 1. *Let Δ be a set of interpreted sentences in a language $\mathcal{L}$ (for which there is a notion of truth simpliciter) closed under the truth functional connectives and $\Box$. Then*

1. *Δ is a theory iff it contains all substitution instances of classical laws and is closed under all classical rules (appropriate for $\mathcal{L}$).*

2. *Δ is a logic iff it is a theory and is closed under the rule of substitution.*

3. *Δ is sound iff every element of Δ is true.*

4. *Δ is a normal logic iff it is a logic and*

[10] In section 3.2 we briefly consider the possibility that $\Box$ could be defined, albeit from other logical constants. It's worth noting that one can raise many of the issues we discuss in this article without making the assumption that $\Box$ is logical (in virtue of being a logical constant, or defined from them), although we have not done so for reasons of space. Bevan (2022) makes same headway in this direction.

(i) Every instance of the schema $\Box(A \to B) \to \Box A \to \Box B$ *belongs to* Δ

(ii) $\Box A \in \Delta$ *whenever* $A \in \Delta$*.*

Our discussion will be organized as follows: in section 3 we will consider the general case where the set of logical truths, Δ, is simply any set of truths closed under classical logic and the rule of substitution—i.e. any sound logic. In section 4 and 5 we consider two more specific accounts of logical truth, one inspired by Bolzano in which a logical truth is a matter of having only true substitution instances, and another by Tarski where one instead quantifies into the position of the non-logical constants.

3 The Metalogical Constraint

We take $\mathcal{L}$ to be an interpreted language containing the unary operator $\Box$; and we take Δ to be a set of sentences of $\mathcal{L}$ – the putative 'logical truths'—meeting the three previous constraints (truth, closure under substitution, and classicality). The general form of the Bridge Principle then looks like this:

The Metalogical Constraint $\Box A$ is true if and only if $A \in \Delta$, for every sentence A of $\mathcal{L}$.

And we may then ask whether there is an interpretation of $\Box$ which satisfies this constraint for some sound logic Δ.

A step in this direction would be to find, for a given logic Δ, a model M of Δ which is such that $\Box A$ is true in M if and only if $A \in \Delta$. For the metalogical constraint would then be satisfied as long as the truth of an interpreted sentence of $\mathcal{L}$ can be identified with the truth of the corresponding un-interpreted sentence in M. The soundness of Δ also follows from this identification and the fact that M is a model of Δ.

3.1 Propositional languages

We start by considering the simplest non-trivial language in which we could formulate this constraint: the language of propositional modal logic. This language is obtained from closing an infinite set of propositional constants $\{p_1, p_2, \ldots\}$ under the binary connective $\wedge$, and the unary connectives $\neg$ and $\Box$. We adopt standard abbreviations for the other logical connectives. A model, in this context, will be identified with a certain sort of *valuation*: a function defined on the language of propositional modal logic that maps sentences to 1 or 0, representing truth and falsity respectively. Clearly a conjunction must be true if and only if both conjuncts are true, and the negation of sentence true if and only if the sentence is false. In keeping with the metalogical interpretation, we must also require that the necessitation of a sentence is true if and only if the sentence is a member of the logic Δ.[11]

Definition 2 (Meta-valuations)**.** *Let Δ be a logic, a set of sentences closed under classical logic and the rule of substitution. A function $v : \mathcal{L} \to \{0, 1\}$ is a Δ-valuation if and only if*

- $v(A \wedge B) = \min(v(A), v(B))$
- $v(\neg A) = 1 - v(A)$
- $v(\Box A) = 1$ *iff* $A \in \Delta$

The metalogical constraint is thus automatically satisfied (relative to a valuation) with respect to a logic, Δ. In virtue of being a logic, Δ is closed under classical logic, and the rule of substitution. Soundness, by contrast, corresponds to a non-trivial condition. It is satisfied by a logic Δ relative to v only if the every sentence in Δ is true in v. We thus introduce the following concept:

Definition 3. *A logic Δ is existentially coherent iff there exists a Δ-valuation v that satisfies Δ (for every $A \in \Delta$, $v(A) = 1$).*

We will drop the word 'existentially' when no ambiguity arises. Intuitively, a logic is coherent when it admits an interpretation of $\Box$ in which it is both sound and satisfies the metalogical constraint; this provides us with the desired model. Not every logic is coherent. Just consider the inconsistent logic consisting of all formulas whatever! Nor is every consistent normal modal logic coherent. For consider the normal modal logic whose sole additional axiom is $\Box p \vee \Box \neg p$. If this logic were sound then it would contain either p or $\neg p$ as a theorem, and so inconsistent after all.

Our chief concern is to figure out what the logic of logical necessity is. We can make a little bit of progress on this question: given the metalogical constraint with respect to a sound logic Δ, it is possible to show that every instance of the the K and T axioms must be true. If we additionally assume that the set of logical truths, Δ, is a *normal modal logic* then every instance of the $\mathsf{4}$ axiom must also be true.

[11] This is essentially the notion of a metavaluation found in Fine (1971–3), pp.1-5. Meyer (1971) contains a similar definition, except that he instead requires that $v(\Box A) = 1$ if and only if $\Box A \in \Delta$.

K $\Box(A \to B) \to \Box A \to \Box B$

T $\Box A \to A$

4 $\Box A \to \Box\Box A$

For K, the truth of $\Box(A \to B)$ and $\Box A$ ensure that $A \to B, A \in \Delta$. Since Δ is a logic, $B \in \Delta$; and so by the metalogical constraint, $\Box B$ is true. For T, suppose that $\Box A$ is true. The metalogical constraint tells us that $A \in \Delta$; and so, by the soundness of Δ, A is true. For $\mathsf{4}$, the truth of $\Box A$ similarly ensures that $A \in \Delta$ and so, by the normality of Δ, it follows that $\Box A \in \Delta$, which by the metalogical constraint means that $\Box\Box A$ is true.

We may obtain a model-theoretic version of this result by replacing 'truth' with 'truth in v' in the above reasoning, and replacing appeals to soundness with appeals to coherence:

Proposition 1. *If Δ is a coherent logic that is sound with respect to a Δ-valuation v, then every instance of the* K *and* T *axioms are true in v. If Δ is a coherent normal modal logic every instance of* $\mathsf{4}$ *is also true in v.*

Remark 1. While the metalogical constraint secures the *truth* of all the instances of K, T and (perhaps) $\mathsf{4}$ for logical necessity, it does not secure their *logical truth*—i.e. their membership in Δ—or indeed their logical necessity, since we have placed very few conditions on Δ apart from being a sound logic. We might take this to motivate positing a further sufficient condition on being a logical truth on top of being a sound logic, expressing a kind of completeness property: that Δ must contain any sentence whose substitution instances are true in all metavaluations for Δ. Adding this further constraint on top of our other constraints will ensure that Δ is normal and indeed an extension of $\mathsf{S4M}$. We discuss an equivalent constraint in section 4.

Our constraints also place some negative restrictions on the logic of logical necessity. John McKinsey argued that the Brouwerian axiom, B, cannot be part of the logic of logical necessity.[12]

B $A \to \Box\Diamond A$

His argument relied on the fact that, given the rule of substitution, claims of the form $\Diamond A$ are rarely logically true unless A is, and so claims of the form $\Box\Diamond A$ are rarely true. We reconstruct this argument in the setting of propositional modal logic below.

[12] See Anderson and Belnap (1975) p123, also recounted in Humberstone (2016) p168.

Proposition 2. *No coherent normal modal logic contains the* B *principle.*

Suppose, for reductio, that Δ is a coherent normal modal logic containing B. If v is a Δ-valuation that satisfies Δ then every instance of B holds in v since $\mathsf{B} \in \Delta$. For a sentence letter, p, $v(p) = 1$ or $v(\neg p) = 1$, so that either $v(\Box\Diamond p) = 1$ or $v(\Box\Diamond\neg p) = 1$. If the former, then $\Diamond p \in \Delta$, and since Δ is closed under the rule of substitution, $\Diamond\bot \in \Delta$. Since Δ is normal, this means it is the inconsistent logic. But as we previously observed every coherent logic is consistent. In the case that $v(\Box\Diamond\neg p) = 1$ we may conclude $\Diamond\neg\top \in \Delta$ using similar reasoning.

Remark 2. There are various ways one could resist this argument against the B principle. Carnap's theory of logical necessity (found in, for example, Carnap (1946), Carnap (1947)) includes all the principles of $\mathsf{S5}$. However Carnap's notion of validity is not closed under the rule of substitution, a key premise in McKinsey's argument; see for instance the discussion in Schurz (2001), Meadows (2012) §3.1 and Cresswell (2013). Further discussion of Carnap's theory of logical necessity can be found in these references, as well as in Hendry and Pokriefka (1985), Cocchiarella and Freund (2008) and the edited volume Rini et al. (2016).[13]

It is instructive to look at some equivalent ways of stating the property of existential coherence. Note that the definition of an existentially coherent logic involves an existential quantifier over Δ-valuations, but it could just as well have involved a a universal quantifier. If the logic Δ satisfies the latter condition—i.e. it holds in all Δ-valuations—we call the logic *universally coherent* (following the notion of coherence defined in Meyer (1971)).

Proposition 3. *A logic* Δ *is existentially coherent if and only if it is universally coherent.*

The argument here, however, relies on features rather distinctive to propositional languages.[14] In contexts where these might be differentiated we will

[13] Particularly relevant to the present section is Cresswell (2013), who isolates Carnap's propositional modal logic via the notion of a Carnapian valuation. These are defined as in definition 2, except for the clause for necessity: $v(\Box A) = 1$ if and only if $u(A) = 1$ for every Carnapian valuation u. A related account of logical necessity, inspired by the Tractatus, is given in Cocchiarella (1974).

[14] The right-to-left direction holds because for any two Δ-valuations, v and u, and non-modal formula A, it is possible to construct a substitution i such that $v(A) = u(iA)$ (by mapping letters to their negations when u and v disagree, leaving them alone otherwise). Because Δ is a logic i is invertible, $A \in \Delta$ if and only if $iA \in \Delta$, so that $v(\Box A) = u(\Box iA)$, and indeed $v(A) = u(iA)$ for arbitrary A. Thus, if $A \in \Delta$, and u is a Δ-valuation in which

distinguish existentially coherent from universally coherent logics (see the subsequent section).[15] Since it is existential coherence that is relevant to the consistency of the metalogical constraint, we will always take 'coherence' to mean 'existential coherence' going forwards.

In the context of normal modal logics we find some other revealing characterizations of coherent logics. Let us say that the maximalization of a logic, Δ, is the theory obtained by adding to Δ the sentences of the form $\Diamond A$ where A is consistent in the logic Δ.[16] The maximalization of Δ intuitively states that every one of the things that are consistent is possible.

Proposition 4. *A normal modal logic Δ is coherent if and only if its maximalization is consistent.*

Remark 3. Note that the maximalization of a logic is a theory but is not itself a logic unless it is inconsistent. The maximalization of any normal logic, if it is consistent, will contain $\Diamond p$ but not $\Diamond \bot$ and so is not closed under the rule of substitution.

Our final characterization of coherent logics relates them to a well-understood property in modal logic, the disjunction property[17]

Definition 4 (The disjunction property)**.** *A normal modal logic Δ has the disjunction property when the following holds for any sentences $A_1, \ldots, A_n$:*

$$\text{If } \Box A_1 \vee \ldots \vee \Box A_n \in \Delta \text{ then } A_k \in \Delta \text{ for some } k \text{ with } 1 \leq k \leq n.$$

Thus we have:

Δ holds, then iA is true in u for any substitution i, and thus A holds in any other valuation v.

[15]There is an issue of specifying what the analogue of a metavaluation is for a language containing the language of predicate logic. However this is resolved, one would expect to be able find valuations that disagree about non-modal logical statements that are independent of first-order logic (for example, whether there are three individuals or not). This means that universally coherent logics cannot contain any "substantive" non-modal logical statements. This is especially acute in the case of higher-order languages, where highly complicated non-modal logical statements can be formulated, such as versions of the continuum hypothesis. Universal coherence also doesn't imply existential coherence: the existence of models of a logic where the truth of $\Box A$ coincides with belonging to the logic is likely highly non-trivial for higher-order logics (for the propositional case existence it is trivial).

[16]A is consistent in Δ iff the smallest theory containing $\Delta \cup \{A\}$ is distinct from $\mathcal{L}$.

[17]See, for example, van Benthem and Humberstone (1983), Chagrov and Zakharyaschev (1997) chapter 15.

Proposition 5. *A normal modal logic is coherent iff it has the extended disjunction property.*[18]

Note that while we have been able to deduce various properties of coherent logics, we have not yet shown that there are any. We end the subsection by describing a general sufficient condition for a logic to be coherent in terms of Kripke frames, and showing that a few well-known normal modal logics are coherent. The reader unfamiliar with the notion of a Kripke frame may skip ahead.

Definition 5. *A pointed Kripke frame is a triple $\mathcal{F} = (W, R, w)$ where W is a set (the worlds), $R \subseteq W \times W$ (the accessibility relation) and $w \in W$ (the root), and w bears the ancestral of R to every element of W. If $v \in W$, we write $\mathcal{F} \uparrow v$ for the pointed frame with root v obtained by restricting W and R to the worlds that v bears the ancestral of R to. We call this a generated submodel of $\mathcal{F}$ by v.*

Every class of pointed frames determines a logic (see, for instance, Goranko and Otto (2007) definition 4). Moreover, this logic is normal whenever that class is additionally closed under generated submodels.

Definition 6 (Disjoint p-morphic copies)**.** *A class $\mathcal{C}$ of pointed frames is closed under disjoint p-morphic copies iff for any pointed frames $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathcal{C}$, there is another pointed frame $\mathcal{F} \in \mathcal{C}$, worlds $w_1, \ldots, w_n$ accessible to the designated world of $\mathcal{F}$, and p-morphisms $f_i : \mathcal{F} \uparrow w_i \to \mathcal{F}_i$ for $i = 1 \ldots n$ such that $w_i \uparrow \cap w_j \uparrow = \emptyset$ when $i \neq j$, and $f_i(w_i)$ is the designated world of $\mathcal{F}_i$.*

Example 1 (Coalesced sums)**.** *Given disjoint pointed Kripke frames $\mathcal{F}_1, \ldots, \mathcal{F}_n$, their coalesced sum is the pointed Kripke frame $\mathcal{F} = (W, R, w_0)$ where:*

- $W := W_1 \cup \ldots \cup W_n \cup \{w_0\}$
- $R := R_1 \cup \ldots \cup R_n \cup (\{w_0\} \times W)$

The designated worlds $w_i \in W_i$, and the identity mappings from $\mathcal{F} \uparrow w_i$ to $\mathcal{F}_i$ comprise the relevant p-morphisms.

Note that many properties of the component frames are inherited by the coalesced sum. For instance, if $\mathcal{F}_1 \ldots \mathcal{F}_n$ are some combination of the properties of being reflexive, serial or transitive, then $\mathcal{F}$ is also that combination of reflexive, serial or transitive. So these classes are all closed under taking disjoint p-morphic copies.

[18] In Bacon and Fine (2023) it is shown that universal coherence (and thus existential coherence) is equivalent to an extension of the disjunction property: that if A_0 is non-modal, and $A_0 \vee \Box A_1 \vee \ldots \vee \Box A_n \in \Delta$ then $A_k \in \Delta$ for some k with $1 \leq k \leq n$. The extended disjunction property is equivalent to the disjunction property: see Jeřábek.

Proposition 6. *If Δ is characterized by a class of pointed frames that are closed under disjoint p-morphic copies, then Δ is coherent.*

Since the following logics are all characterized by the class of all pointed frames satisfying some combination of reflexivity, seriality or transitivity, we can immediately infer:

Corollary 7. *K, KT, S4, KD, K4D are coherent.*

Less work has been done on non-normal modal logics, although one case is particularly straightforward. Let Γ be the set of theorems of classical logic in $\mathcal{L}$ — the substitution-instances of tautologies. Γ is a natural logic to plug into the metalogical constraint since it is the smallest logic. It is obvious that every substitution instance of a tautology is true in any Γ-valuation, since valuations assign classical truth values with respect to the truth functional connectives. Thus:

Proposition 8. *Classical logic, Γ, is coherent.*

This interpretation of $\Box$ is explored in Urquhart (2010), where $\Box A$ is read as 'it's a tautology that A'.

We conclude this section with a question. Say that a coherent logic, Λ, is maximal iff whenever $\Lambda \subseteq \Delta$ and Δ is coherent, $\Delta = \Lambda$. It follows by Zorn's lemma that there are maximal coherent logics, and indeed we will encounter an example of one later (Medvedev logic). What are other examples of maximal coherent logics, and does the class of maximal coherent logics have a simple characterization?

3.2 The Metalogical Constraint in Other Languages

Apart from the logic, Δ, the other parameter we can vary is the language $\mathcal{L}$. In this section we briefly discuss the the situation for languages that extend propositional languages with quantificational resources. We will focus on higher-order languages, as they contain other quantificational languages, such as first-order languages, as fragments. Expressions of a language can generally be divided into different logical types: sentences, predicates, connectives, names and so on. Higher-order languages contain, in addition to the truth functional connectives, quantifiers that can bind into the position of different logical types analogous to the way that first-order quantifiers bind into the position of a name. The non-logical constants may also include constants of other types; propositional letters being the special case of a non-logical constant of sentence type. As before, logical necessity is treated as an operator expression,

$\Box$, which we can introduce as either a new primitive logical constant, or by definition from other logical constants.[19]

For non-propositional languages the notion of a model of a logic Δ which satisfies the metalogical constraint will be more involved than the notion of a metavaluation from the propositional case. But we can circumvent talking about models by focusing on either of the two equivalent formulations of coherence given in propositions 4 and 5.[20] Since having a consistent maximalization and the disjunction property are straightforwardly equivalent, even in this context, it does not matter which we use:[21]

Definition 7. *Let $\mathcal{L}$ be a higher-order language containing an operator constant $\Box$. A higher-order logic Δ is maximalizable iff its maximalization, the theory obtained by adding $\{\Diamond A \mid \neg A \notin \Delta,\ A \text{ closed}\}$ to Δ, is consistent.*

Some results on maximalizable higher-order logics can be found in Bacon (2023) §18.5-6 and Bacon and Dorr (forthcoming) appendix E. The main tool for establishing these results is a generalization of the coalesced sum construction presented in the previous section.

4 The Substitutional Constraint

In the previous section we imposed a very minimal constraint on the logical truths—we required only that they constitute a sound logic. This constraint is fairly liberal: classical propositional logic, for instance, is a sound logic, but it intuitively doesn't include *all* the logical truths as there are presumably distinctively modal logical truths. We noticed, for instance, that every substitution instance of K and T are true in any Γ-valuation (Proposition 1) when Γ is classical propositional logic. One might therefore think that these are

[19]Because of the expressive limitations of propositional languages, the possibility of defining $\Box$ was not open in that setting. The crucial assumption here is that $\Box$ can be defined without reference to non-logical constants so that it is left alone by substitutions of a language.

[20]So we here exclude from consideration non-normal logics.

[21]Note that a normal modal logic Δ has the disjunction property iff it has a consistent maximalization. In particular if $A_1, \ldots, A_n$ are individually consistent with Δ, $\Diamond A_1, \ldots, \Diamond A_n$ are jointly consistent with Δ. For if $\Box\neg A_1 \vee \ldots \vee \Box\neg A_n \in \Delta$, then by the disjunction property one of the A_i is not consistent with Δ after all. The consistency of the maximalization of Δ follows by compactness (recalling that we defined a set X to be consistent iff no theory containing $X \cup \Delta$ is $\mathcal{L}$, where a theory is a set of sentences closed under the classical laws). It is straightforward to show that a coherent logic has the disjunction property.

plausibly logical truths even though they are not theorems of classical propositional logic. In this section, and the next, we will consider strengthening the metalogical constraint with completeness conditions along these lines.

We begin with an account, going back to Bolzano, in which a sentence is logically true if it is true in virtue of its logical form alone.[22] The logical form of a sentence is that which is common to every substitution instance of that sentence. So according to this account, a sentence may be taken to be logically true if and only if every one of its substitution instances is true. Now clearly every substitution instance of a theorem of classical propositional logic is true, and so is a logical truth by this account. Indeed, the set of all sentences with only true substitution instances is a sound logic—it is closed under the rule of substitution, and consists only of truths—and so this account of logical truth meets the minimal set of requirements from section 2. But classical propositional logic is far from complete. As noted above, every instance of K and T can be shown to be true, and so K and T will count as logical truths too; something we were unable to conclude from the metalogical constraint alone. Thus the strengthened constraint is:

The Substitutional Constraint $\Box A$ is true if and only every substitution instance of A is true.

One direction of this constraint holds of any operator satisfying the more general metalogical constraint, for the truth of $\Box A$ implies that $A \in \Delta$; but since Δ is closed under substitutions, and contains only truths, every substitution instance of A must be true. However, the other direction is distinctive to this theory. This account of necessity was originally investigated in McKinsey (1945).

The substitutional constraint tells us a lot more about the logic of logical necessity. We will see that all instances of K, T, $\mathsf{4}$, M and a principle we call Sub are true, and thus are also logical truths.[23] We are similarly able to show that the logical truths are closed under necessitation. For if A is a logical truth, i.e. every substitution instance of A is true, then so is every substitution instance of any substitution instance of A. This ensures that any substitution instance of $\Box A$ is true, so that $\Box A$ is also a logical truth. Together this means that the logic of logical necessity must include every theorem of $\mathsf{S4MSub}$.

[22]See, for instance, Morscher (2018) section 4.8. Bolzano's account was directed at propositions, not sentences, and so could be considered a direct account of logical necessity that does not pass through the notion of logical truth.

[23]To establish the truth any instance of K, for example, assume that $\Box(A \to B)$ and $\Box A$ are true, so that every substitution instance of $(A \to B)$ and A are true. This means that every substitution instance of B is true and finally that $\Box B$ is true, securing K.

4.1 Propositional languages

We again focus on the case of propositional languages and, as before, we introduce the notion of a valuation which will behave classically with respect to $\wedge$ and $\neg$, and in which $\Box$ will be governed by the substitutional constraint.

Definition 8 (Substitutional valuation)**.** *A function $v : \mathcal{L} \to \{0,1\}$ is a substitutional valuation if and only if*

- $v(A \wedge B) = \min(v(A), v(B))$
- $v(\neg A) = 1 - v(A)$
- $v(\Box A) = 1$ *if and only if* $v(A') = 1$ *whenever* A' *is a substitution instance of* A*.*

In this section only, 'valuation' will refer to substitutional valuations. Following the Bolzanoean characterization of logical truth, validity is defined by:

Definition 9. *A sentence A is valid iff all substitution instances of A are true in every valuation: $v(A') = 1$ for every valuation v and substitution instance A' of A.*

We stated above that given the substitutional constraint the logic of logical necessity extends S4M. This is the smallest normal modal logic containing T, 4, encountered above, and McKinsey's axiom M:[24]

M $\Box\Diamond A \to \Diamond\Box A$

Proposition 9. *Every theorem of* S4M *is valid.*

It is a relatively easy exercise to establish that validity is closed under the rule of necessitation (if A is valid, so is $\Box A$) and that the K, T, 4 and M axioms are valid. To illustrate the general mode of argument, we establish the validity of M. Suppose that v is a valuation and A an arbitrary sentence. If $v(\Box\Diamond A) = 1$ then for every substitution i, there exists a substitution j such that $v(j(iA)) = 1$ (writing iB for the result of applying a substitution to a sentence B). Let i be any substitution that maps every letter to a tautology, $\top$. It can be shown that $v(j(iA)) = v(iA)$ for any substitution j. It follows that $v(iA) = 1$, since by assumption we know there is a j with $v(j(iA)) = 1$. Similarly, it follows that for *any* j, $v(j(iA)) = 1$, so $v(\Box iA) = 1$ and finally $v(\Diamond\Box A) = 1$.

[24]Relative to transitive reflexive frames, McKinsey's axiom characterizes the class frames in which every world sees a world that only sees itself.

A more complex example of a validity is a principle we will call the 'subset principle', Sub (for reasons that will become clear). Let us write $Z \subseteq^0 Y$ ($Z \subset^0 Y$) to mean that Z is a non-empty (proper) subset of Y. To each set, $X = \{1, \ldots, n\}$, associate in some canonical way some consistent propositional formulas, $A_1, \ldots, A_n$, that are pairwise inconsistent and have a tautologous disjunction. For every non-empty $Y \subseteq X$, we will define a formula D_Y^X (or simply D_Y when X is clear from context). When $|Y| = 1$, $Y = \{m\}$ for some m, so set $D_{\{m\}} = \Diamond\Box A_m \wedge \bigwedge_{k \neq m} \neg\Diamond\Box A_k$. When D_Z is defined for $|Z| \leq k$ and $|Y| = k+1$, set $D_Y = \bigwedge_{Z \subset^0 Y} \Diamond D_Z \wedge \Box(\bigwedge_{Z \subset^0 Y} \Diamond D_Z \vee \bigvee_{Z \subset^0 Y} D_Z)$. The subset principle is then:[25]

Sub $\bigvee_{Y \subseteq^0 X} D_Y$

Proposition 10. Sub *is valid.*

The proof of this proposition is more involved; it can be found in Bacon and Fine (2023) proposition 30.

The reader may have noted that we have not yet provided any examples of substitutional valuations. In fact establishing the existence of a valuation is quite non-trivial, and appeared in Harvey Friedman's *One hundred and two problems in mathematical logic* (see Friedman (1975), problem 42), along with the question of whether the valuation was unique once the truth values of the propositional letters has been fixed. The difficulty arises because the clause for the truth of $\Box A$ (in a valuation) depends on the truth of all substitution instances of A. Because these may have a much higher complexity than A—they may even involve $\Box A$ itself—there is a degree of circularity here. The existence portion of the conjecture was independently settled affirmatively by Prucnal (1979) and Fine (later published in Bacon and Fine (2023)), and the uniqueness portion remains open.

The Prucnal-Fine valuation can be described as a Δ-valuation (in the sense of section 3) with respect to a suitable choice of logic, Δ. Consider the class of Kripke frames of the form $(P(X) \setminus \emptyset, \supseteq)$ where X is a finite set—i.e. finite partial orders of non-empty sets under the superset ordering. Call the logic of these frames Med.

Proposition 11 (Prucnal, Fine)**.** *Every* Med*-valuation is a substitutional valuation.*

[25] It can be seen to be imply an axiom schema from Holliday (2017) and Hamkins et al. (2015), where $1 < k \leq m$:

$$(\bigwedge_{i \leq m} \Diamond\Box A_i \wedge \neg\Diamond \bigvee_{i \neq j} A_i \wedge A_j) \rightarrow \Diamond(\bigwedge_{i \leq k-1} \Diamond\Box A_i \wedge \bigwedge_{k \leq j \leq m} \neg\Diamond\Box A_j)$$

Sub is strictly stronger than this axiom in the presence of S4.

This fact puts further bounds on the logic of logical necessity. For if A is valid, then every substitution instance of A is true in any substitutional valuation, and thus in any Med-valuation. Since Med-valuations are substitutional valuations, this means $\Box A$ is true in any Med-valuation, which only happens if $A \in \mathsf{Med}$. Thus if Δ is the set of validities with respect to the substitutional constraint, $\mathsf{S4MSub} \subseteq \Delta \subseteq \mathsf{Med}$.

Yet further progress can be made if the uniqueness conjecture is settled affirmatively:

Conjecture 12. *For any two substitutional valuations, v and u, if $v(p) = u(p)$ for every propositional letter p, $v = u$.*

If this conjecture is true, the logic of logical necessity must be exactly Med: for if $A \in \mathsf{Med}$, then $\Box A$ is true in every Med-valuation, and thus in every substitutional valuation. This implies that every substitution instance of A is true in every valuation, and so A is valid.

There are several open questions concerning the logic Med. It is unknown, for instance, whether it is recursively axiomatizable.[26] If it turned out that $\mathsf{Med} = \mathsf{S4MSub}$, this would provide an alternative route to determining Med as the logic of logical necessity since,as we observed above, the validities are sandwiched between these two logics.

4.2 Restricted substitution classes

Say that a set of substitutions, S, is a substitution class if it contains the trivial substitution (mapping each sentence letter to itself), and is closed under compositions of substitutions. Several early discussions, including McKinsey's original paper, focused on interpreting $\Box$ in terms of a restricted substitution class.[27] When S is a substitution class, an S-valuation is then a function $v : \mathcal{L} \to \{0, 1\}$ that behaves classically with respect to the truth-functional connectives, $\wedge$ and $\neg$ (see definition 8) and is such that $v(\Box A) = 1$ if and only if $v(iA) = 1$ for every substitution $i \in S$.

By limiting S to substitutions that replace letters with non-modal formula, for instance, the problem of circularity discussed in the previous section is avoided. Using the same sort reasoning, McKinsey was able to show that every theorem of $\mathsf{S4}$ is true in any S-valuation. However, this status does not extend to M or Sub. In in establishing the validity of M and Sub in the

[26] It is not finitely axiomatizable; see Shehtman (1990).

[27] This may have been due to the difficulty of establishing the existence of any substitutional valuations. See the discussion of p.169 of Humberstone (2016).

full substitution class one must appeal to special features of that class—in the former case, for instance, we appealed to the existence of a substitution that mapped every letter to a tautology. Drake (1962) establishes that S4 is in fact complete with respect to arbitrary S-valuations.

Various substitution classes have been studied. Humberstone (2016) considers a propositional modal language with primitive logical constants $\top$ and $\bot$, and restricts attention to the set of substitutions that map each letter to itself, $\top$ or $\bot$. Another natural substitution class consists of the substitutions that map letters to $\Box$-free formulas. In Bacon and Fine (2023), the logic of various non-modal substitution classes are investigated and shown to be equivalent to the logic of certain classes of Kripke frames. The two logics mentioned above, for instance, can be distinguished by the presence (in the former case) or absence (in the latter case) of the principle:

Grz $\Box(\Box(A \to \Box A) \to A) \to A$

4.3 Other languages

While we have focused on propositional languages, it is possible to formulate the substitutional constraint in richer languages.[28] McKinsey's original examples involved sentences in subject predicate form, and suggested that he was thinking of substitutions as acting on these subsentential components. In this setting the status of the necessity of identity and distinctness is somewhat more subtle; and more discussion of these issues can be found in §6 of Bacon and Fine (2023) and in §7.3 of Bevan (2022).

5 The Quantificational Constraint

In the early twentieth century, logicians began to employ a quantificational account of logical truth. For instance, Bernays and Schönfinkel (1928) spell out the notion of logical truth and logical consistency for first-order languages in terms of truth *simpliciter* in a higher-order language: the logical truth of $\forall x(Fx \vee \neg Fx)$ for them amounts to the mere truth of $\forall F \forall x(Fx \vee \neg Fx)$—'for any property and any individual, that individual has the property or it does not'—and the logical consistency of $\forall xy(Rxy \to Ryx)$ amounts to the truth of $\exists R \forall xy(Rxy \to Ryx)$.[29] Tarski later turned this into an explicit definition

[28]The substitutional constraint in the first-order context is discussed in Bacon and Fine (2023) section 6, and in Bevan (2022) section 7.3.

[29]See Bernays and Schönfinkel (1928), p347. This is essentially the same notion used in Hilbert and Ackermann (1928)—the first modern logic textbook—although they also

of logical truth, any given instance of which is equivalent to the proposal in Bernays and Schönfinkel.[30]

Given a language $\mathcal{L}$, we can consider a higher-order language, $\mathcal{L}^\forall$, extending $\mathcal{L}$ with higher-order quantifiers, $\forall_\sigma$, that can bind into the position of each type σ had by some non-logical constant. The logical truth of a sentence $A(c_1, \ldots, c_n)$, where $c_1 \ldots c_n$ enumerate all of the non-logical constants of various types, $\sigma_1, \ldots, \sigma_n$, appearing in A, amounts to the truth of $\forall_{\sigma_1} x_1 \ldots \forall_{\sigma_n} x_n A(x_1, \ldots, x_n)$, where each x_i is a variable of the same logical type as c_i, namely σ_i. So the bridge principle now reads as follows:[31]

The Quantificational Constraint $\Box A(c_1 \ldots c_n)$ is true if and only if $\forall x_1 \ldots x_n.A(x_1, \ldots, x_n)$ is true

Note that in the case where $\mathcal{L}$ is already a full higher-order language, both the original sentence and its universalization belong to $\mathcal{L}$, so that we can formulate this as an object language biconditional:

$$\Box A(c_1 \ldots c_n) \leftrightarrow \forall x_1 \ldots x_n.A(x_1, \ldots, x_n)$$

The construction of a model of this biconditional can be illustrated well-enough in the simpler setting of propositionally quantified modal propositional logic.[32] In this setting the constraint has the form $\Box A(p_1, \ldots, p_n) \leftrightarrow \forall x_1 \ldots x_n.A(x_1 \ldots x_n)$, where the p_i are propositional letters, and the x_i are propositional variables. Consider the transitive reflexive Kripke frame consisting of an infinite tree where each world has a countably infinite number of immediate successors. We can interpret propositionally quantified modal logic in this frame by interpreting the letters as sets of worlds, and letting the propositional quantifiers range over arbitrary sets of worlds (see Fine (1970)). Suppose

required that the first-order quantifiers be restricted by a predicate representing the domain of quantification, which then also gets quantified out in the definitions of logical truth and consistency. This way of defining logical truth features prominently in Williamson (2013), under the label 'metaphysical universality'.

[30] According to Tarski (1937) logical truth may be reduced to satisfaction: a sentence, '$A(c_1, .., c_n)$', containing only the non-logical constants $c_1, \ldots, c_n$, is a logical truth if and only if '$A(x_1, \ldots, x_n)$' is satisfied by all assignments of values to $x_1 \ldots x_n$ (this will require higher-order quantification unless the non-logical constants are all singular terms). In Tarski (1931) Tarski also showed that satisfaction for formulas of any finite-order fragment of higher-order logic can be reduced to pure higher-order logic.

[31] A related treatment of logical necessity is considered in Cocchiarella (1975) §4; there the quantificational definition of logical truth is employed as a means of recursively paraphrasing away the notion of logical necessity into a higher-order logic without any modal operators.

[32] A proof that this constraint is satisfiable in a full higher-order language is sketched in Bacon (2020); the relevant lemmas are shown in Bacon (2023) chapter 18. The simpler construction described here is from Bacon and Fine (2023) section 5.

that $A_1, A_2, \ldots$ enumerate the formulas satisfiable in this frame, $M_1, M_2, \ldots$ the interpretations of the letters that witness the satisfiability of these sentences, and finally suppose we have similarly enumerated the immediate successors of the root world $w_1, w_2, \ldots$. We can make the formula A_n true at w_n, since the worlds accessible to w_n under their natural order forms an isomorphic copy of the whole frame, and so we can copy the interpretation of the letters found in M_n on to this subframe. Using this strategy we can make every sentence satisfiable in the whole frame true in some immediate successor of the the root world. In the resulting model, A is satisfiable at the root of the frame, if and only if $\Diamond A$ is true at the root. Because the propositional quantifiers range over all sets of worlds, we also have that A is satisfiable if and only if $\exists x_1 \ldots x_n.A(x_1, \ldots, x_n)$ is true at the root, securing the dualized version of the quantificational constraint.

Much remains to be done. One outstanding question concerns which logics the quantificational constraint is compatible with. The above model demonstrates that one such logic is the logic of the infinite tree described above, with respect to an unrestricted interpretation of the propositional quantifiers. Another natural question concerns whether there is a substitutional interpretation of the higher-order quantifiers under which the quantificational constraint is satisfied; this would provide an interpretation in which the quantificational and substitutional constraints from section 4 are jointly satisfied.[33]

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[33]For some discussion of some of the wider implications of this question, see section 13.2 Bacon (2023) and the discussion surrounding conjecture 13.1.

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