Appendices

Anon*

May 13, 2024

A Preliminaries

A.1 Language

We work in a simply typed higher-order modal language: there are two base types, e and t, and given any types σ and τ there is a functional type $(\sigma \to \tau)$. We omit type brackets when they are associated to the right, and will write '$M : \sigma$' as short for 'M is a term of type σ' or 'M, of type σ,'.

The terms of language are defined as follows. For each type σ, there will be infinitely many variables of that type. We typically represent these with upper and lower case letters towards the end of the latin alphabet, like X, Y, Z and x, y, z. Occasionally we will use more suggestive names like 'suc' and 'add' for variables depending on their function. Whenever M is a term of type $\sigma \to \tau$ and N a term of type σ, (MN) is a term of type τ and whenever M is a term of type τ and x a variable of type σ, $(\lambda x.M)$ is a term of type $\sigma \to \tau$. Finally we have primitive terms for the logical constants: $\forall_\sigma : (\sigma \to t) \to t$, $\to: t \to t \to t$, and $\Box : t \to t$. We may introduce $\exists_\sigma, \bot, \wedge, \vee, \leftrightarrow, =_\sigma$ as abbreviations in any of the standard ways. For instance, $\bot$ may be identified with $\forall_{(t\to t)\to t}\forall_t$, $=_\sigma$ with $\lambda xy\forall_\sigma X(Xx \to Xy)$.

We adopt some further conventions, following **?**. We adopt infix notation for the binary logical connectives and identity. λs immediately following a quantifier are omitted. Given a term $P : \sigma \to t$ we write $\forall^P_\sigma$ for $\lambda X\forall_\sigma x(Px \to Xx)$, and $\exists^P_\sigma$ for $\lambda X\exists_\sigma x(Px \wedge Xx)$. We use $\vec{x}$ for sequences $x_1...x_n$. $\lambda\vec{x}, \forall\vec{x}$ etc. stand for strings of λs or quantifiers — e.g. the first amounts to $\lambda x_1\lambda x_2 \ldots$

*...

— and $R\vec{x}$ stands for $Rx_1 \dots x_n$. $\vec{\sigma} \to \tau$ stands for $\sigma_1 \to \sigma_2 \to \dots \to \tau$. $M[N/x]$ is the result of replacing every free occurrence of v in M with N provided no free variable in N becomes bound.

The languages we consider may contain further non-logical constants. As usual logics and theories will be identified with sets of terms of type t.

A.2 Formalizing mathematical notions in higher-order logic

$\Diamond_{\vec{z}} := \lambda R \lambda \vec{z}. \neg \Box \neg R\vec{z}$	$\subseteq_{\vec{\sigma}} := \lambda XY \forall_{\vec{\sigma}} \vec{z} (X\vec{z} \to Y\vec{z})$
$\sim_{\vec{\sigma}} := \lambda XY.(X \subseteq_{\vec{\sigma}} Y \wedge Y \subseteq_{\vec{\sigma}} X)$	$\leq_{\vec{\sigma}} := \lambda XY. \Box X \subseteq_{\vec{\sigma}} Y$
$\mathrm{Rig}_{\vec{\sigma}} := \lambda X \Box \forall_{\vec{\sigma} \to t} Y (\Box \forall^X_{\vec{\sigma}} \vec{z}. Y\vec{z} \leftrightarrow \forall^X_{\vec{\sigma}} \vec{z}. \Box Y \vec{z})$	$\mathrm{World}_{\vec{\sigma}} := \lambda R (\Diamond_{\vec{\sigma}} R \wedge \forall S (R \leq_{\vec{\sigma}} S \vee R \leq_{\vec{\sigma}} \neg_{\vec{\sigma}} S)$
$\mathrm{Ub}^{\preceq} := \lambda Xy. \forall z (Xz \to z \preceq y)$	$\mathrm{Lub}^{\preceq} := \lambda Xy.\, \mathrm{ub}\, Xy \wedge \forall z (\mathrm{ub}\, Xz \to y \preceq z)$
$\mathrm{Dom}_\sigma := \lambda Rx. \exists_\sigma y. (Rxy \vee Ryx)$	$\mathrm{Trans}_\sigma := \lambda R \forall_\sigma xyz (Rxy \wedge Ryz \to Rxz)$
$\mathrm{Ancest}_\sigma := \lambda Sxy \forall R (\mathrm{Trans}\, R \wedge S \subseteq_\sigma R \to Rxy)$	$\mathrm{Fun}_\sigma := \lambda S \forall_\sigma xyy' (Sxy \wedge Sxy' \to y =_\sigma y')$
$F : X \to Y := \forall^X x \exists^Y ! y. Fxy$	$F : X \xrightarrow{1-1} Y := \forall^X xx'y. (Fxy \wedge Fx'y \to x = x')$
$\mathrm{PO} := \lambda PR. PR$ is a partial order	$\mathrm{Lattice} := \lambda PR. P, R$ is a lattice
$\mathrm{Compl} := \lambda PR. PR$ is a complemented lattice	$\mathrm{Dist} := \lambda PR. PR$ is a distributive lattice
$\mathrm{BA}_\sigma := \lambda PR. PR$ is a Boolean algebra	$\mathrm{CBA}_\sigma := \lambda PR. PR$ is a complete Boolean algebra

Table 1: Abbreviations

In this section we show how to formalize various familiar mathematical notions in higher-order logic. For the sake of readibility definitions will be given in ordinary English, and we will only provide explicit definitions in the language of higher-order logic when the required definition is not obvious.

We begin with some order-theoretic notions. A *partial order* at type σ consists of a property, $P : \sigma \to t$, and a relation $\preceq : \sigma \to \sigma \to t$ which is transitive, reflexive and antisymmetric with respect to the type σ entities satisfying P. P entities are called *elements* in the partial order. A partial order $P, \preceq$ has meets and joins when any two elements have a greatest lower bound and a least upper bound in the partial order, in which case we call $P, \preceq$ a *lattice*. A lattice is *complete* when for any property F there is a greatest greatest lower bound and least upper bound of the Fs in P. We will sometimes write $a \sqcap b$ and $a \sqcup b$ for the (unique) meet and join of a and b: note that in using this notation we are not treating $\sqcap$ itself as a $\sigma \to \sigma \to \sigma$ term—rather $a \sqcap b$ is a syntactically simple term introduced by existential instantiation. A lattice is *distributive* when $a \sqcap (b \sqcup c)$ and $(a \sqcap b) \sqcup (a \sqcap c)$ are the same. A Boolean algebra $P, \preceq$ is a *complemented* distributive lattice:

for every element, a, there is another element b such that $a \sqcup b$ is the greatest element of the lattice and $a\sqcap$ is the least element. A *well-order* at type σ is total partial order such for every property $F : \sigma \to t$, if there are any F elements, there is a $\preceq$-least F element. The ancestral of a relation R holds between x and y when every transitive relation extending R holds between x and y ($\text{Ancestral} := \lambda Sxy\forall R(\text{Trans}\, R \wedge S \subseteq_\sigma R \to Rxy)$).

Given terms $F : \sigma \to \tau \to t$, and $X : \sigma \to t, Y : \tau \to t$, we write $F : X \to Y$ to mean that F is a functional relation between X and Y: every X bears F to a unique Y. $F : X \xrightarrow{1-1} Y$ means that this relation is one-one: no two Xs bear F to the same Y, and $F : X \xrightarrow{\text{onto}} Y$ means that it is onto: for any Y there is some X that bears F to that Y, and $F : X \xrightarrow{\text{bij}} Y$ if it is both one-one and onto. We use '**P**' to stand for a sequence of variables '$P : \sigma \to t, \preceq_P : \sigma \to \sigma \to t$' and '**Q**' for '$Q : \tau \to t, \preceq_Q : \tau \to \tau \to t$'. If **P** and **Q** are partial orders, then we write $\mathbf{P} \cong \mathbf{Q}$ iff the partial orders are isomorphic: there exists $F : P \xrightarrow{\text{bij}} Q$ such that for any whenever Fxx' and Fyy', $x \preceq_P y$ if and only if $x' \preceq_Q y'$.

A *natural number structure* at type σ consists of an entity $0 : \sigma$, and a functional one-one relation $\text{suc} : \sigma \to \sigma \to t$ such that: nothing bears suc to 0, and moreover, any relation with 0 in its field that relates x toy y when x is in its field and $\text{suc}\, xy$, contains suc: $\forall R(\text{Dom}\, Rz \wedge \forall x(\text{Dom}\, Rx \wedge \text{suc}\, xy \to Rxy) \to \forall xy(\text{suc}\, xy \to Rxy))$. A *first-order natural number structure* consists of the above, and additionally relations $+, \times, <$ such that .

$$+ := \lambda nmk.\forall R(Rn0n \wedge \forall ii'jj'(\text{suc}\, ii' \wedge \text{suc}\, jj' \wedge Rnij \to Rni'j') \to Rnmk)$$

$$\times := \lambda nmk.\forall R(Rn00 \wedge \forall ii'jj'(\text{suc}\, ii' \wedge \text{add}\, njj' \wedge Rnij \to Rni'j') \to Rnmk)$$

$$<:= \lambda nm.\forall R(\forall ij(\text{suc}\, ij \to Rij) \wedge \forall ijk(Rij \wedge Rjk \to Rik) \to Rnm)$$

The *domain* of a natural number structure is the field of $<$. We will write **N** to abbreviate a sequence of variables $z : \sigma, S : \sigma \to \sigma \to t$ and we write $\text{Nat}^\sigma\, \mathbf{N}$ for the statement that z and S together form a natural number structure at type σ; the same notation will be adopted for first-order natural number structures.

A *real number structure* at a type σ consists of a total partial order property $R : \sigma \to t, \preceq$, elements $0, 1 : \sigma$ and ternary relations $+, \times : \sigma \to \sigma \to \sigma \to t$ that are functional with domain R representing addition and multiplication. We will write $x + y$ as short for the description for the unique z such that $+xyz$, and similarly for $\times$. Addition and multiplication are commutative and associative and distributive in the sense that $x \times (y+z) = (x \times y) + (x \times z)$. 0 and 1 are the units of $+$ and $\times$ respectively (e.g. $\forall_\sigma x(+x0y \to x = y)$, and every element of R has an additive inverse and every element apart from 0 has a multiplicative inverse—i.e. for each x there is a y such that $x+y=0$ and for each $x \neq 0$ there is a y such that $x \times y = 1$. Moreover if $x \preceq y$ then $x+z \preceq y+z$ and if $0 \leq x$ and $0 \leq y$, $0 \leq x \times y$. Finally it is complete: for any property of elements F that has an upperbound in R has a least upperbound. A *first-order real number structure* consists of the preceding along with a predicate N such that

$$N := \lambda x.\forall F(F0 \wedge \forall y(Fy \wedge \forall z(+x1z \to Fz) \to Fx)$$

We will write $\mathbf{R}$ for a sequence of variables $R, N, +, \times, 0, 1, <$ of the appropriate types. We write $\mathrm{Real}^\sigma\, \mathbf{R}$ to say that they form a real number structure.

Next some modal notions. A proposition, property or relation P of type $\vec{\sigma} \to t$ is possible$_{\vec{\sigma}}$ when it is possible that there exist entities $\vec{x}$ that instantiate P; P is necessary in the dual case. We say that P entails Q, when $\lambda\vec{z}(R\vec{z} \to S\vec{z})$ is necessary$_{\vec{\sigma}}$. A world proposition (property, relation) is something that is possible, and such that, for any other proposition (property, relation), it entails it or its negation. A property (relation) X is rigid iff the X restricted quantifiers necessarily satisfy the Barcan formula and its converse: $\mathrm{Rig}_{\vec{\sigma}} := \lambda X \Box \forall_{\vec{\sigma} \to t} Y(\Box \forall^X_{\vec{\sigma}} \vec{z}.Y\vec{z} \leftrightarrow \forall^X_{\vec{\sigma}} \vec{z}.\Box Y\vec{z})$.

Quantification over "natural number structures" is strictly speaking a sequence of universal quantifiers, one for each element of the signature of a natural number structure. Thus we will need some notation for representing such sequences.

- We use $\mathbf{N}$ for a sequence of variables with the following types $0 : \sigma, \mathrm{suc} : \sigma \to \sigma \to t$.
- We use $\mathbf{R}$ for a sequence of variables with the following variables $R, N : \sigma \to t, +, \times : \sigma \to \sigma \to \sigma \to t, 0, 1 : \sigma, <: \sigma \to \sigma \to t,$.
- We use $\mathbb{N}$ for the *canonical* natural number structure: the sequence of terms $\mathrm{NumQuant}, 0^Q, \mathrm{suc}_Q, <_Q, +_Q, \times_Q$ defined above.
- We use $\mathbb{R}$ for the canonical real number structure: sequence of terms given in theorem A.2 below.

PC $\vdash A$ whenever A is a tautology.

UI $\vdash \forall_\sigma F \to Fa$.

β $A[(\lambda x.M)N] \leftrightarrow A[M[N/x]]$.

η $A[\lambda x.(Fx)] \leftrightarrow A[F]$, where x is not free in F.

K $\Box(A \to B) \to \Box A \to \Box B$

MP If $\vdash P$ and $\vdash P \to Q$, then $\vdash Q$.

Gen If $\vdash P \to Q$, and v is not free in P, $\vdash P \to \forall v Q$.

Nec If $\vdash A$ then $\vdash \Box A$

Figure 1: The Background Logic, $\mathsf{H}^\Box$

RC $\forall_{\vec{\sigma}\to t} R \exists_{\vec{\sigma}\to t} X.(\mathrm{Rig}\, X \wedge R \sim_{\vec{\sigma}} X)$

B $\Box \forall_t p(p \to \Box\Diamond p)$

LB $\forall_{\vec{\sigma}\to t} P(\Diamond_{\vec{\sigma}} P \leftrightarrow \exists_{\vec{\sigma}\to t} W.(\mathrm{World}_{\vec{\sigma}}\, W \wedge W \leq_{\vec{\sigma}} P))$

MP $\forall \mathbf{B}(\mathrm{SmallCBA}_\sigma\, \mathbf{B} \to \exists_t P.(\mathbf{B} \cong \mathbf{P} \wedge P \neq_t \top))$

Figure 2: Key Modal Principles

A.3 Logical systems

Here we state some logics of interest. The minimal system $\mathsf{H}^\Box$ is presented in figure 1. We adopt the usual notation from modal logic for modal principles: $\mathsf{T} := \Box\forall_t p(\Box p \to p), \mathsf{4} := \Box\forall_t p(\Box p \to \Box\Box p)$ and $\mathsf{B} := \Box\forall_t p(p \to \Box\Diamond p)$.

To $\mathsf{H}^\Box$ we can add further principles, listed in figure 2, which we denote by appending their names separated by a dot—e.g. $\mathsf{H}^\Box.5$ for adding $\mathsf{T}, \mathsf{4}$ and B, $\mathsf{H}^\Box.5.\mathsf{RC}$ including RC, etc.

In the statement of MP, $\mathbf{B}$ stands for a pair of variables $B : \sigma, \preceq : \sigma \to \sigma \to t$ and $\mathbf{P}$ for $P : t, \leq$, recalling that $\leq$ is defined as $\lambda pq.p \wedge q = p$. SmallCBA is the property of being a complete Boolean algebra whose cardinality is no bigger than the number of propositions.

Throughout we will appeal to couple of facts that may be derived in these systems about the existence of natural and real number structures. First we define what we will call the *canonical natural number structure*, consisting of the cardinality quantifiers:

$$\begin{aligned}
0_Q &:= \lambda X.\top \\
\mathrm{suc}_Q &:= \lambda Q\lambda X.\exists y.(Xy \wedge Q\lambda z(Xz \wedge z \neq y) \\
\mathrm{NumQuant} &:= \lambda Q\forall Z((Z0 \wedge \forall P(ZP \to Z(\mathrm{suc}\, P)) \to ZQ) \\
<_{\mathrm{Q}} &:= \lambda PQ\forall Z(Z0(\mathrm{suc}\, 0)\wedge \\
&\qquad \forall P'Q'(ZP'Q' \to (ZP'\,\mathrm{suc}\, Q' \wedge Z\,\mathrm{suc}\, P'\,\mathrm{suc}\, Q')) \to ZPQ) \\
+_{\mathrm{Q}} &\coloneqq \lambda xyz.\forall R(\forall w(Rw0w \wedge \forall uv(Rwuv \to Rw(\mathrm{suc}\, u)(\mathrm{suc}\, v))) \to Rxyz) \\
\times_{\mathrm{Q}} &\coloneqq \lambda xyz.\forall R(\forall w(Rw00 \wedge \forall uv(Rwuv \to Rw(\mathrm{suc}\, u)(\mathrm{add}\, vw))) \to Rxyz)
\end{aligned}$$

Let the axiom of potential infinity be the following principle:[1]

Potential Infinity $\forall Q(\mathrm{NumQuant}\, Q \to \Diamond Q(\lambda x.\top))$

Theorem A.1. *Given the axiom of Potential Infinity (in $\mathsf{H}^\Box$), the canonical natural number structure is indeed a natural number structure.*

Second we will appeal to the fact that, given the axiom of Potential Infinity, and one of several auxiliary assumptions, there exists a real number structure that can be constructed from the canonical natural number structure, and consists of properties of finite numerical quantifiers. We call this the *canonical real number structure*. The definition of this structure, and the proof that it is a real number structure is rather involved. It exploits the non-obvious, but well-known, fact that you can define operations on the powerset of natural numbers that turns it into a into a complete ordered field.

There is a slight wrinkle with transposing the set-theoretic construction to the higher-order framework: sets, unlike properties, are individuated extensionally. We cannot, then, straightforwardly identify reals with properties of naturals since there would be many coextensive properties corresponding to any given real. There are several work arounds. If we have Rigid Comprehension, we can identify reals with rigid properties of naturals, since these *are* individuated extensionally. Without Rigid Comprehension we don't have any guarantee that there are enough rigid properties to play the role of all

[1] cf. **?**, **?**.

the reals. However, if we have the well-ordering principle or some similar choice principle we can instead pick a particular property from in a given equivalence class of coextensive properties to be a representative of a given real.

The Well-ordering Principle $\exists R.\,\mathrm{WO}\,R \wedge \mathrm{Dom}\,R \sim \lambda x.\top$

Thus we have:

Theorem A.2. *Given the axiom of Potential Infinity, and either Rigid Comprehension or the Well-Ordering Principle (in* $\mathsf{H}^{\Box}$*), it is possible to construct a real number structure at the type* $\sigma \to t$*, where* $\sigma = (e \to t) \to t$ *the type of quantifiers. Moreover, it is possible to do so in such a way that every property of canonical natural numbers is coextensive with exactly one element of the real number structure.*

There is one final work around that requires no additional assumptions beyond Potential Infinity. We can define a quasi-real number structure as consisting of the same data as a real number structure with the addition of an equivalence relation $\approx$ to represent identity: so that we have $R, N : \sigma \to t$, $+, \times : \sigma \to \sigma \to \sigma \to t$, $0, 1 : \sigma$, $<, \approx: \sigma \to \sigma \to t$. We then require the operations $+, <, \times, R, N$ to respect the notion of identity in the sense that, e.g., if $+abc$ and $a \approx a', b \approx b', c \approx c'$ then $+a'b'c'$. We also modify the conditions for being a complete ordered field by substituting all occurrences of $=$ with $\approx$, so that, for instance, the commutativity law becomes $+abc \wedge +bac' \to c \approx c'$. The notion of an isomorphism between quasi-real number structure is now a (possibly non-functional) relation which preserves $\approx$ and the other field operations. Quasi-real number structures can be constructed, without additional assumptions, from properties of canonical natural numbers using coextensiveness as the notion of identity. Note that every real number structure is automatically a quasi-real number structure with $\approx := =_\sigma$. Appeals to theorem A.2 can be substituted to appeals to the existence of quasi-real number structures in this paper, but in the contexts we need canonical real number structures we will always either have Rigid Comprehension or a well-ordering available.

B Consistency proofs

B.1 Model of Classicism with first-order arithmetical contingency

Here we prove

Theorem B.1. *There is a first-order arithmetical sentence, $A(0, \text{suc}, <, \text{add}, \text{mult})$ and a model of Classicism (H with $\mathsf{S4}$ and Intensionalism) which is structurally contingent. I.e. the model makes*

$$\Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge A(\mathbf{X})) \wedge \Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge \neg A(\mathbf{X}))$$

true.

The proof uses methods described in **?**. There is described a class of "modal models" which is sound and complete with respect to Classicism. Among these are models are "extensionally full" models, which have, for every subset of their domain, a property that has that subset as its extension, and satisfies similar conditions for relations. Extensionally full models with an infinite type e domain are arithmetically standard in the following sense.

Definition B.1. *M is arithmetically standard iff $M \models \forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to A(\mathbf{X}))$ if and only if $A(0, \text{suc}, +, \times, <)$ is an arithmetical truth.*

Here we use the expression $M \models A$ to mean that the sentence A is true in the model M. We have, by Proposition 18.7 and Corollary 18.4 **?** the following fact:

Theorem B.2. *Given any set of modal models, $\mathcal{C}$, there is an arithmetically standard modal model M such that, whenever $N \in \mathcal{C}$, $N \models A$ where A is closed, $M \models \Diamond A$.*

We may construct a model of first-order arithmetical contingency as follows. Let us first find an arithmetical truth, A, whose structural translation, $\forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to A(\mathbf{X}))$, cannot be derived in Classicism. The consistency statement for Classicism would do. By the completeness theorem there is a modal model N of $\exists\mathbf{X}(\text{Nat}\,\mathbf{X} \wedge \neg A(\mathbf{X}))$. Let $\mathcal{C} = \{N\}$: by theorem B.2 above there is an arithmetically standard model M such that $M \models \Diamond\exists\mathbf{X}(\text{Nat}\,\mathbf{X} \wedge \neg A(\mathbf{X}))$. Moreover $M \models \exists\mathbf{X}(\text{Nat}\,\mathbf{X} \wedge A(\mathbf{X}))$. For an arithmetically standard model must make $\exists\mathbf{X}\,\text{Nat}\,\mathbf{X}$—$\bot$ is not an arithmetical truth, so $M \not\models \forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to \bot)$—and $\forall\mathbf{X}(\text{Nat}\,\mathbf{X} \to A(\mathbf{X}))$ since A is an arithmetical truth. M is a model of $\Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge A(\mathbf{X})) \wedge \Diamond\exists\mathbf{X}(\text{Nat}(\mathbf{X}) \wedge \neg A(\mathbf{X}))$ as required.

B.2 Model of Classicism, **S5**, **RC** and first-order analytic contingency

We would like to construct a model of the following two claims:

$$\Diamond\exists\mathbf{R}(\mathrm{Real}\,\mathbf{R}\wedge\mathrm{CH}\,\mathbf{R})$$

$$\Diamond\exists\mathbf{R}(\mathrm{Real}\,\mathbf{R}\wedge\neg\,\mathrm{CH}\,\mathbf{R})$$

where CH is:

$$\lambda\mathbf{R}(\forall X\subseteq R(\exists F: R\xrightarrow{1-1}X\vee\exists F: X\xrightarrow{1-1}N))$$

Here $\mathbf{R}$ is short for the sequence of variables $R, N, 0, 1, \mathrm{add}, \mathrm{mult}, <$, with R a unary predicate representing the reals of the structure and N representing the naturals.

Below we construct a set-theoretic model, in a background of ZFC+CH, and offer a sketch of proof that it satisfies the desired properties. Create a full functional model as follows.

- $\mathbb{P}$:= the disjoint sum of the partial order $(\{p\mid p$ is a finite partial function from $\omega_2\times\omega$ to $2\},\supseteq)$ and $(\{@\},\{(@,@)\})$.
- $\mathbb{B} := RO(\mathbb{P})$, the regular open subsets of $\mathbb{P}$.
- $D^t=\mathbb{B}\times 2$.
- $D^e=\omega$
- $D^{\sigma\to\tau}={D^{\tau}}^{D^{\sigma}}$
- $\forall_\sigma$ given by meet in the Boolean algebra, similarly for the logical connectives.

$\mathbb{B}$ is a complete Boolean algebra. Intuitively it consists of a solitary atom, $\{@\}$—which will serve as our actual world—and then a large atomless false proposition $P:=\mathbb{P}\setminus\{@\}$. We will show that according to this model "there exists a real structure in which CH true" is true at the actual world, but false throughout the atomless portion of logical space. We will use $\bigsqcap$ and $\bigsqcup$ to denote the meets and joins of elements in this algebra, and p^c for the complement of p. Observe that $\mathbb{B}$ has the countable chain condition: every set of consistent pairwise incompatible elements in $\mathbb{B}$ is countable.

The meanings of terms are computed relative to variable assignments g, which map each variable of type σ to an element of D^σ:

- $[\![x]\!]^g = g(x)$
- $[\![MN]\!]^g = [\![M]\!]^g([\![N]\!]^g)$
- $[\![\lambda x.M]\!]^g = a \mapsto [\![M]\!]^{g[a/x]}$
- $[\![\forall_\sigma]\!] = f \mapsto \prod_{a \in D^\sigma} f(a)$
- $[\![\to]\!] = p \mapsto q \mapsto (p^c \sqcup q)$

A formula A is satisfied by g iff $@ \in [\![A]\!]^g$.

We assume, for simplicity, we are working in a language with a constant of type σ for every element of D^σ. If we also play loose with use and mention (or let the elements of D^σ be their own names), this eliminates various bits of fussing involving variable assignments—we can write $[\![A(a_1, ..., a_n)]\!]$ where $a_i \in D^{\sigma_i}$ instead of $[\![A(x_1, ..., x_n)]\!]^g$ where g is a variable assignment mapping x_i to a_i.

Next we appeal to theorem A.2 which guarantees that, given the well-ordering principle, we can construct a canonical real number structure whose elements are properties of natural numbers, and which includes at least one such property with any given extension on the natural numbers. In the present setting we get the following.

Lemma B.1. *Suppose $[\![r \text{ is a well-ordering}]\!] = \top$, and let $\sigma = e \to t$. Then there exists terms, $\mathbb{R} := R, N : \sigma \to t, 0, 1 : \sigma, +, \times : \sigma \to \sigma \to \sigma \to t, <: \sigma \to \sigma \to t$ each in a single parameter r, corresponding to reals, naturals, operations of addition and multiplication, 0, 1 such that $[\![\mathrm{Real}\,\mathbb{R}]\!] = \top$ and $[\![\forall_{e\to t} X \exists_{e \to t} Y(RY \wedge X \sim Y)]\!] = \top$.*

In order for this construction to work we need to check that such an r exists:

Lemma B.2. *The Well-Ordering Principle, WO^σ, is necessarily true in M. Indeed, there is a particular element of the model, $r \in D^{\sigma\to\sigma\to t}$, such that $[\![r \text{ is a well-order}]\!] = \top$.*

Proof. It is sufficient to find a relation, $r \in D^{\sigma\to\sigma\to t}$, such that the semantic value of "r totally orders type σ and is well-founded" in M is $\top$.

Let $<$ be some well-order on D^σ, we may define r as

$$r(a)(b) = \begin{cases} \top & \text{if } a < b \\ \bot & \text{otherwise} \end{cases}$$

It is easily seen that $[\![r \text{ is a total order}]\!] = \top$. It remains to show that $[\![r \text{ is well-founded}]\!] = \top$. It suffices to show $[\![\exists y.fy]\!] \leq [\![\exists y.y \text{ is } f \text{ and } r\text{-minimal}]\!]$ for every $f \in D^{\sigma \to t}$.

Let $f \in D^{\sigma \to t}$. If $[\![\exists y.fy]\!] = \bot$ we are done. If $[\![\exists y.fy]\!] \neq \bot$, it suffices to show that for every $b \leq [\![\exists y.fy]\!]$ there is some $a \in D^\sigma$ such that $[\![a \text{ is } f \text{ and } r\text{-minimal}]\!] \sqcap b \neq \bot$.

Since $[\![\exists y.fy]\!] \neq \bot$, there exists a $d \in D^\sigma$ with $[\![fd]\!] \sqcap b \neq \bot$. Let a be a $<$-minimal element with this feature. Suppose $\bot < b' \leq f(a) \sqcap b$, and $d \in D^\sigma$ with $b' \leq [\![fd]\!]$. Then $[\![fd]\!] \sqcap b \neq \bot$ so $r(a)(d) = \top$ or $a = d$ by the minimality of a, so $[\![rad \vee a = d]\!] = \top$. Since $d \in D^\sigma$ was arbitrary, $f(a) \sqcap b \leq [\![\forall_\sigma y(fy \to ray \vee a = y)]\!]$. This means $f(a) \sqcap b \leq [\![a \ f \text{ and } r\text{-minimal}]\!] \sqcap b \neq \bot$ as required. $\square$

First we show that $\Diamond \exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \text{CH}\,\mathbf{R})$ is true in the model. Indeed $\exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \text{CH}\,\mathbf{R})$ is true in the model (i.e. holds at @) for we know from lemma B.1 that there are elements of the model, $\mathbb{R}$, such that $[\![\text{Real}\,\mathbb{R}]\!] = \top$. But it can also be shown that that the truth of CH is "absolute" in the model.

Lemma B.3. $\exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \text{CH}\,\mathbf{R})$ *is true in M if and only if the continuum hypothesis is true.*

M is extensionally full in the sense of **?** appendix A4: for any subset $X \subseteq D^\sigma$ there is an element $f \in D^{\sigma \to t}$ such that for all $a \in D^\sigma$, $@ \in [\![fa]\!]$ if and only if $a \in X$. Thus in extensional contexts quantification over properties in the model is equivalent to quantification over sets in the metalanguage. This can be used to show that counterexamples to the higher-order version of CH in the model would be counterexamples to the set-theoretic continuum hypothesis and conversely.

Next we show that $\Diamond \exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \neg\,\text{CH}\,\mathbf{R})$ is true in the model. In fact $P \leq [\![\exists \mathbf{R}(\text{Real}\,\mathbf{R} \wedge \neg\,\text{CH}\,\mathbf{R})]\!]$ where P is the atomless portion of logical space, $\mathbb{P} \setminus \{@\}$.

Lemma B.4. *The the semantic value of "there is a real number structure $R, \ldots$ at type $e \to t$ and an uncountable property of those reals which the reals cannot inject into" is the worldless portion of logical space P.*

Proof. Our strategy is to use lemma B.1 to find a real number structure $\mathbb{R} = R, N, \ldots$ made of properties of natural numbers, and then show that P entails that it does not satisfy CH.

For each $\alpha < \omega_2$ we define we define some special properties of natural numbers, $a_\alpha \in D^{e\to t}$, as follows

$$a_\alpha(x) = \{p \in \mathbb{P} \mid p(\alpha, x) = 1\}$$

Intuitively the a_α are highly contingent properties of natural numbers that are nonetheless necessarily coextensive with some R property, and necessarily no pair of them are coextensive.

Now to define the counterexample to the continuum hypothesis, G. In the worldless regions of logical space, G is uncountable and R cannot be injected into G. $G : D^{e\to t} \to D^t$

$$G(a) = \begin{cases} \top & \text{if } a = a_\beta \text{ for some } \beta < \omega_1 \\ \bot & \text{otherwise} \end{cases}$$

Now, let $c' \in D^{(e\to t)\to(e\to t)\to t}$ be the relation necessarily relates each property (element of $D^{\sigma\to t}$) to the minimal such element coextensive with it, obtained from lemma B.1 (i.e. c' with $[\![c'$ is a choice relation for $\sim]\!] = \top$). We can define $c(a)(b) = [\![a \sim b]\!]$ when $b = a_\alpha$ for some α and $= c'ab$ otherwise— it is easily seen that $[\![c$ is a choice relation for $\sim]\!] \geq P$. By lemma B.1 we have a real structure R, in the parameter c such that $[\![\forall X \exists Y(X \sim Y \wedge RY)]\!] = \top$ and $[\![Ra_\alpha]\!] = \top$ for every $\alpha < \omega_2$.

We first show that for any $g \in D^{(e\to t)\to(e\to t)\to t}$, $[\![g : R \xrightarrow{1-1} G]\!] \subseteq \{@\}$—i.e. g is not injective from R to G at the worldless portion of space. Suppose otherwise, for contradiction. So $b := [\![g : R \xrightarrow{1-1} G]\!]) \sqcap P > \bot$ (we add the conjunct so that we can effectively ignore what g is like at the only world in the algebra). Using the axiom of choice, we may define a function $f : \omega_2 \to \omega_1$ that maps each $\alpha < \omega_2$ to a β which *might* enumerate a real number that is G.

$$f(\alpha) = \beta \text{ where } [\![ga_\alpha a_\beta]\!] \sqcap b > \bot$$

We first show that $f : \omega_2 \to \omega_1$, as claimed. Since $b \leq [\![ga_\alpha a_\beta \to Ga_\beta]\!]$ (i.e. b contains the claim that g has codomain G), and since $Ga_\beta = \bot$ when $\beta \geq \omega_1$, $b \leq [\![\neg ga_\alpha a_\beta]\!]$ when $\beta \geq \omega_1$, i.e. $[\![ga_\alpha a_\beta]\!] \sqcap b = \bot$ and so no $\beta \geq \omega_1$ is in the range of f. Thus $f : \omega_2 \to \omega_1$.

Now pick some $\gamma < \omega_1$ such that $f^{-1}(\gamma)$ is uncountable. There must be such a γ since $\omega_2 > \omega_1$. Now consider the following set:

$$\{[\![ga_\alpha a_\gamma]\!] \sqcap b \mid f(\alpha) = \gamma\}$$

The elements of this set are all non-zero (by the definition of f), pairwise incompatible (by the fact that $b \leq [\![g \text{ is injective}]\!]$), and uncountable by our choice of γ. We then have an uncountable anti-chain in $\mathbb{B}$ which is not possible.

To show that $[\![G \text{ is uncountable}]\!]$ we use a similar strategy, this time finding an injective $f : \omega_1 \to \omega$ for the contradiction.

$\square$

C Derivations

C.1 Proof that there is no first-order arithmetical contingency in $\mathsf{H}^{\square}.5$

Proposition C.1 (Prior)**.** *The necessity of distinctness, and the Barcan and converse Barcan formulas at any type are derivable in* $\mathsf{H}^{\square}.5$*.*

The first is proved in **?** pp.206-207. Essentially if $\Diamond x = y$ then, by $\square$(the Necessity of Identity) we can infer $\Diamond\square x = y$ from which we obtain $x = y$. The necessity of distinctness follows from the contrapositive of $\Diamond x = y \to x = y$. The second result is also due to Prior–see **?**.

Under the assumption that there is a natural number structure of individuals, the finite numerical quantifiers form a natural number structure with respect to the following definitions

Proposition C.2. *In* $\mathsf{H}^{\square}.5$ *we can derive the following*

1. *Being a numerical quantifier,* NumQuant*, is rigid.*
2. *The relations* $<_Q, +_Q, \times_Q$ *on the numerical quantifiers are rigid.*

Proof. In $\mathsf{S5}$, rigidity of a relation R is equivalent to showing (i) $\forall\vec{x}(R\vec{x} \to \square R\vec{x})$. For (i) implies (ii) $\forall\vec{x}(\neg R\vec{x} \to \square\neg R\vec{x})$, and we can establish rigidity as follows. For any relation Z, we have by the Barcan and converse Barcan formulas $\square\forall\vec{x}(R\vec{x} \to Z\vec{x}) \leftrightarrow \forall\vec{x}\square(R\vec{x} \to Z\vec{x})$. But given (i), and the K axiom, the right-hand-side implies $\forall\vec{x}(R\vec{x} \to \square Z\vec{x})$. And given (ii), $\forall\vec{x}(R\vec{x} \to \square Z\vec{x})$ implies the right-hand-side. Thus $\square\forall\vec{x}(R\vec{x} \to Z\vec{x}) \leftrightarrow \forall\vec{x}(R\vec{x} \to \square Z\vec{x})$. We can then apply universal generalization and necessitation to this argument, to obtain the statement that R is rigid.

So now we prove that every numerical quantifier is necessarily a numerical quantifier by induction. Let Z be the property of necessarily being a numerical quantifier: $\lambda Q.\Box\,\mathrm{NumQuant}\,Q$. We will show that Z applies to 0 and is closed under suc. From the definition of a numerical quantifier that every numerical quantifier has Z.

For the base case note that it is a trivial logical truth that every property that applies to 0 and is closed under suc applies to 0, so this logical truth is necessary. Thus we have $\Box\,\mathrm{Quant}\,0$.

For the inductive step, assume that ZQ, i.e. Q is necessarily a numerical quantifier. It follows that it's necessary any property that applies to 0 and is closed under suc applies to $\mathrm{suc}\,Q$; i.e. it's necessary that $\mathrm{suc}\,Q$ is a numerical quantifier.

The proof of the rigidity of $<_Q, +_Q$ and $\times_Q$ are similar. For the case of $<$, the base case consists in showing $\Box 0 < \mathrm{suc}\,0$ and the inductive step, that if $\Box Q < P$ then also $\Box\,\mathrm{suc}\,Q < \mathrm{suc}\,P$ and $\Box Q < \mathrm{suc}\,P$.

□

Lemma C.1. *In* $\mathsf{H}^{\Box}.5$*, we can prove* $\forall\vec{Q}(\mathrm{NumQuant}\,\vec{Q} \wedge A^{\dagger} \rightarrow \Box A^{\dagger})$ *and* $\forall\vec{Q}(\mathrm{NumQuant}\,\vec{Q} \wedge \neg A^{\dagger} \rightarrow \Box\neg A^{\dagger})$ *for any first-order arithmetical sentence* A*.*

Proof. We prove this by induction on first-order arithmetical sentences. The base cases $Q = P$ and $Q < P$ follow by propositions C.2 and C.1.

The inductive cases for the truth functional connectives are straightforward. The quantificational case follows from the rigidity of NumQuant. □

Lemma C.2. *In* $\mathsf{H}^{\Box}.5$ *if there is a natural number structure of individuals, then, necessarily,* $0, <$ *is a natural number structure on the numerical quantifiers.*

Theorem C.1. *In* $\mathsf{H}^{\Box}.5$*, there is no structural arithmetical contingency:*

$$\forall\mathbf{N}(\mathrm{Nat}(\mathbf{N}) \wedge A(\mathbf{N}) \rightarrow \Box\forall Ry(\mathrm{Nat}(\mathbf{N}) \rightarrow A(\mathbf{N})))$$

where $A(<, 0)$ *is an arithmetical sentence.*

Proof. Suppose that $\mathbf{N}$ is a natural number structure and $A(\mathbf{N})$. Since there is a natural number structure, the axiom of Potential Infinity holds, so we know that the canonical number structure $\mathbb{N}$, consisting of numerical quantifiers, forms a natural number structure. Since $A(\mathbf{N})$, $A^{\dagger}(\mathbb{N})$ by Dedekind's theorem. So by Lemma C.1 $\Box A^\dagger(\mathbb{N})$. Since, given S5, the axiom of potential infinity is necessarily true if true at all, $\mathbb{N}$ is necessarily natural number structure. So we know that necessarily any natural number structure $\mathbf{N}$ at type e will be isomorphic to $\mathbb{N}$ and also make $A(\mathbf{N})$ true. □

C.2 Proof of no second-order analytic contingency given □RC and LB

Recall that we use $\mathbf{R}$ as short for a sequence of variables $R, N : \sigma \to t, +, \times : \sigma \to \sigma \to \sigma \to t, 0 : \sigma, 1 : \sigma$. We will write '$x$ is an element of the structure $\mathbf{R}$' in the exposition to mean Rx.

Here will prove the following theorem.

Theorem C.2. *In* $\mathsf{H}^\Box.\Box\mathsf{RC.LB}$ *one can derive all instances of The Necessity of Analysis in the language of second-order analysis. Whenever A is a sentence of second-order analysis:*

$$\Box\forall\mathbf{X}(\text{Real}\,\mathbf{X} \to A(\mathbf{X})) \vee \Box\forall\mathbf{X}(\text{Real}\,\mathbf{X} \to \neg A(\mathbf{X}))$$

The formulas of second-order logic (relative to type σ) are defined as follows

- The formulas $Xy_1 \ldots y_n$ are second-order analytical formulas when $x, y, z : \sigma$ and $X : \sigma \to \ldots \to \sigma \to t$.
- If A and B are second-order then $A \wedge B$ and $\neg A$ are too.
- If A is second-order, then $\forall x(Rx \to A)$ is too.
- If A is second-order, then $\forall X(\forall\vec{x}(X\vec{x} \to \bigwedge_i Rx_i) \wedge \text{Rig}\,X \to A)$ is to.

Note that there is a copy of second-order logic for any choice of σ, although it is typically assumed that $\sigma = e$. Given a choice of variables $\mathbf{R} = R, N : \sigma \to t, +, \times : \sigma \to \sigma \to \sigma \to t, 0 : \sigma, 1 : \sigma$, we say that a formula is a formula of second-order analysis iff it is second-order and $\mathbf{R}$ appear free, and is a sentence of second-order analysis iff its free variables are exactly $\mathbf{R}$.

Observe that the second-order quantifiers are restricted to rigid properties. This is in line with standard mathematical practice, which treats second-order logic as extensional. However, in the presence of Rigid Comprehension, one could drop the restriction to rigid properties without making a difference to the truth of any formula of second-order analysis. A straightforward induction shows that formulas of second-order analysis cannot distinguish between coextensive properties:

Proposition C.3 (Analytic Extensionality)**.** *In* $\mathsf{H}^{\Box}$ *one can derive* $\forall\vec{z}(X\vec{z} \leftrightarrow Y\vec{z}) \rightarrow A \rightarrow A[Y/X]$ *for any second-order analytical formula* A

This fact does not extend to arbitrary formulas of higher-orderese, since in the full language one can formulate intensional notions, such as property identity, which are not part of the language of second-order analysis.

First, a few remarks on the proof strategy. A more straightforward version of the proof to follow is possible if we make the assumption of the necessity of distinctness. First show that the rigidification, $\mathbf{R}$, of any real number structure, obtained by rigidifying R, N, $+$, $\times$ and $<$, is necessarily a real number structure. Then we can show, using the Leibniz Biconditionals, that any sentence about the reals that is true in a rigid real number structure is necessarily true in that structure. It follows by Huntington's theorem that, necessarily, any real number structure is isomorphic to $\mathbf{R}$, and so makes true anything that $\mathbf{R}$ actually makes true.

In the absence of the necessity of distinctness, a rigid real number structure could fail to be a real number structure (if, say, everything in its domain became identical). It will be convenient to use a restricted notion of necessity in this argument defined as

$$\Box_{\mathbb{N}} := \lambda p.\Box(\Diamond\exists\mathbf{N}.\,\mathrm{Nat}\,\mathbf{N} \rightarrow p)$$

Using 'necessary', 'possible', 'rigid', 'inflexible' and so on in this new sense, the rigidification of the canonical real number structure will be inflexible due to the fact that it is built out of numerical quantifiers which are $\Box_{\mathbb{N}}$-necessarily distinct. Now we can show that any given real number structure is isomorphic to an inflexible real number structure (the canonical reals), and then proceed as above. We will call a structure $\mathbf{R}$ rigid when $R, N, +, \times, <$ are rigid, and inflexible when additionally, $\forall xy(Rx \rightarrow \Box_{\mathbb{N}} x \neq y)$, here defining these modal concepts in terms of $\Box_{\mathbb{N}}$. Note, also, that if $\mathbf{R}$ is rigid with respect to $\Box$ it is also rigid with respect to $\Box_{\mathbb{N}}$, so that Rigid Comprehension implies the variant of that principle involving $\Box_{\mathbb{N}}$.

Once we have shown that the rigidification of the canonical real number structure is inflexible, we show that inflexible real number structures are necessarily complete, and consequently that they are necessarily real number structures (it is of course necessarily an ordered field). This will involve the Leibniz biconditionals.

First, we will need a consequence of Huntington's theorem, that no two real number structures (potentially at different types) can disagree about the truth of second-order analytic statements.

Lemma C.3. *For any second-order analytic statement, A,* $\Box\forall_\sigma\mathbf{R}\forall_\tau\mathbf{S}(\mathrm{Real}^\sigma(\mathbf{R})\wedge \mathrm{Real}^\tau(\mathbf{S}) \rightarrow (A^\sigma(\mathbf{R}) \leftrightarrow A^\tau(\mathbf{S}))$

Here A^σ and A^τ are obtained by shifting which type is playing the role of "first-order" variables to σ. We omit the proof. Note that, like Analytic Extensionality, this theorem does not extend to arbitrary formulas, such as those involving intensional notions, second-order identity or third-order quantification. For instance, one real number structure may consist of necessarily distinct elements, while an isomorphic one might not; second-order identity and third-order quantification allow one to construct similar examples.

Next we need to construct an inflexible real number structure—note that we require only inflexibility with respect to $\Box_\mathbb{N}$. We will use the canonical real number structure obtained from theorem A.2, where we identify reals with rigid properties of the canonical natural number structure (using Rigid Comprehension). The result of rigidifying this structure we will call $\mathbf{R} = R, N, +_\mathbf{R}, \times_\mathbf{R}, 0_\mathbf{R}, 1_\mathbf{R}, <_\mathbf{R}$.

While this structure is clearly rigid, it needs to be shown that it is inflexible and $\Box_\mathbb{N}$-necessarily a real number structure. (Note that the canonical real number structure itself is $\Box_\mathbb{N}$-necessarily a real number structure, by applying theorem A.2 and the fact that the axiom of Potential Infinity is $\Box_\mathbb{N}$-necessary, but we don't know that the canonical real number structure is rigid.) Why is it inflexible? Because the numerical quantifiers are necessarily distinct with respect to $\Box_\mathbb{N}$ the reals—rigid properties of numerical quantifiers—will also be necessarily distinct in the same sense. Of course, without the assumption of Potential Infinity, the numerical quantifiers may not in fact form a natural number structure, and $\mathbf{R}$ may not be a real number structure. Thus we should have:

Lemma C.4. *If the axiom of Potential Infinity holds, then* $\mathbf{R}$ *is an inflexible real number structure.*

Note that if there could have been a real number structure then the axiom of Potential Infinity is true, and if there couldn't have been a real number structure the necessity of analysis holds vacuously. While the above lemma doesn't use Rigid Comprehension, we needed it in our definition of **R**. Next we show that **R** is necessarily a real number structure.

Lemma C.5 (Leibniz Biconditionals)**.** *If the axiom of Potential Infinity holds, then* **R** *is* $\Box_{\mathbb{N}}$*-necessarily a real number structure.*

Proof. We first show that if R is a rigid property of necessarily distinct individuals, $\forall xy(Rx \wedge Ry \wedge x \neq y \rightarrow \Box_{\mathbb{N}} x \neq y)$, then for any element z or **R**, $\lambda x.z < x$ and $\lambda x.x < z$ are rigid. Suppose $\Diamond_{\mathbb{N}}\exists x(z' < x \wedge Fx)$. So $\Diamond_{\mathbb{N}}\exists xz'(x < z' \wedge z = z' \wedge Fx)$, which by rigidity implies $\exists xz'.x < z' \wedge \Diamond_{\mathbb{N}}(z = z' \wedge Fx)$. Finally, by the necessity of distinctness, $z' = z$, so $\exists x.x < z \wedge \Diamond_{\mathbb{N}} Fx$. For the other direction, we know that if for some x, $z < x \wedge \Diamond_{\mathbb{N}} Fx$ then it's necessary that $z < x$ by rigidity, so $\Diamond_{\mathbb{N}}(z < x \wedge Fx)$, and so also $\Diamond_{\mathbb{N}}\exists x(z < x \wedge Fx)$ applying existential generalization under $\Diamond_{\mathbb{N}}$. This reasoning is easily necessitated establishing the rigidity of $\lambda x.z < x$. The other case is shown similarly.

Let P be the higher-order property of being a collection or reals that has no least upperbound:

$$P := \lambda X.(X \subseteq R \wedge \neg\exists y.\, \mathrm{lub}\, yX)$$

Suppose, for contradiction, that **R** is possibly not complete, that is: $\Diamond_{\mathbb{N}}\exists_{e\to t}X.PX$. By the Leibniz biconditionals there is a world property W, that entails P. Now we may consider the property of being a real x such that W entails applying to x—informally, x would have fallen into the unique W collection of properties if W had been instantiated.

$$Y := \lambda x.\Box_{\mathbb{N}}\forall X(WX \rightarrow Xx)$$

By the rigidity of $\leq$, Y consists of only reals (if Yx, W entails $\lambda X(Xx \wedge PX)$, to so x is possibly an $\leq$-real—i.e. stands in $\leq$ to something—and so by rigidity there is something it $\leq$s). By the actual completeness of **R**, Y has a least upperbound, z. We will show that necessarily, z is the least upper bound of the unique property of reals X that has W, if it exists.

First, we establish that z necessarily an upperbound any X that is W. $\Box_{\mathbb{N}}\forall X(WX \rightarrow z \geq X)$ writing $z \geq X$ for $\forall x(Xx \rightarrow z \geq x)$. Suppose otherwise, for contradiction: $\Diamond_{\mathbb{N}}\exists X(WX \wedge \exists x(Xx \wedge x > z))$. Applying some logic inside $\Diamond_{\mathbb{N}}$, $\Diamond_{\mathbb{N}}\exists x > z(\exists X(WX \wedge Xx)$. Applying the rigidity of $\lambda x.x > z$

we get $\exists z > x \Diamond_{\mathbb{N}} \exists X(Wx \wedge Xx)$. Since W is a world property, it cannot be consistent with Xx unless it entails it: so $\Box_{\mathbb{N}} \forall X(WX \to Xx)$ which means Yx by definition of Y. The fact that $x > z$ contradicts the assumption that z is an upperbound of Y.

Next we establish that necessarily z is the least upperbound of the X that is W, when such an X exists. $\Box_{\mathbb{N}} \forall X(WX \to \mathrm{lub}\, zX)$. Suppose for contradiction that $\Diamond_{\mathbb{N}} \exists X(WX \wedge \exists x \geq X.x < z))$. Applying logic under $\Diamond_{\mathbb{N}}$, $\Diamond_{\mathbb{N}} \exists x > z \exists X(WX \wedge x \geq X)$. By the rigidity of $\lambda x.x < z$, we have $\exists x < z \Diamond_{\mathbb{N}} \exists X(WX \wedge x \geq X)$. To complete the contradiction it is sufficient to show that $x \geq Y$, contradicting the assumption that z was the *least* upperbound. So suppose Yy, which means $\Box_{\mathbb{N}} \forall X(WX \to Xy)$. It follows, using the normality of $\Box_{\mathbb{N}}$, that $\Diamond_{\mathbb{N}} x \geq y$. Given the necessity of distinctness, we can infer that in fact $x \geq y$ (for otherwise $x \leq y$ and $x \neq y$, and these must be necessary given the rigidity of $\leq$ and the necessity of distinctness, which is incompatible with $\Diamond_{\mathbb{N}} x \leq y$). Thus $x \geq Y$.

□

Lemma C.6 (Rigid Comprehension)**.** *Let W be a world property of type $(\sigma \to t) \to t$, and $Z : \sigma \to t$ a rigid property. Then if it possible that W is instantiated by a rigid property $\subseteq Z$, then there is an actual rigid property that could have been identical to the W property:*

$$\Box_{\mathbb{N}} \forall Y(WY \to (\mathrm{Rig}\, Y \wedge Y \subseteq Z)) \to \exists X(\mathrm{Rig}\, X \wedge X \subseteq Z \wedge \Box_{\mathbb{N}} \forall Y(WY \to Y = X)$$

Proof. Suppose that $\Box_{\mathbb{N}} \forall Y(WY \to (\mathrm{Rig}\, Y \wedge Y \subseteq Z))$, and Z is the rigid property given by the assumption. Let X be the rigid property coextensive with $\lambda x.(Zx \wedge \Box_{\mathbb{N}} \forall Y(WY \to Yx)$. Clearly X is necessarily rigid, and $X \subseteq Z$. It suffices to show that W entails being coextensive with X, since W entails rigidity and coextensive rigid properties are identical. There are two inclusions to show.

In order to show that $\Box_{\mathbb{N}} \forall Y(WY \to X \subseteq Y)$ it suffices to show

$$\forall x(Xx \to \Box_{\mathbb{N}} \forall Y(WY \to Yx))$$

since by the rigidity of X, we can conclude $\Box_{\mathbb{N}} \forall Y(WY \to \forall x(Xx \to Yx))$. So let x be an arbitrary X. By the definition of X it follows that that $\Box_{\mathbb{N}} \forall Y(WY \to Yx)$, so the claim follows.

In order to show that $\Box_{\mathbb{N}} \forall Y(WY \to Y \subseteq X)$ it suffices to show

$$\forall x(Zx \to \Box_{\mathbb{N}} \forall Y(WY \to Yx \to Xx)$$

since by the rigidity of Z, we can conclude $\Box_{\mathbb{N}}\forall Y(WY \to \forall x(Zx \to Yx \to Xx))$, which is equivalent to the desired claim, since $\Box_{\mathbb{N}}\forall Y(WY \to \forall x(Yx \to Zx))$. So let x be an arbitrary Z. In the case that x is X, we also have $\Box_{\mathbb{N}}Xx$ by rigidity of X, delivering the desired result, $\Box_{\mathbb{N}}\forall Y(WY \to Yx \to Xx)$. In the case that x is not X, that means $\neg\Box_{\mathbb{N}}\forall Y(WY \to Yx)$ or $\neg Zx$. In fact, the first disjunct must be true, for if $\Box_{\mathbb{N}}\forall Y(WY \to Yx)$ but $\neg Zx$ we have have $\Diamond_{\mathbb{N}}Zx$ since $\Box_{\mathbb{N}}\forall Y(WY \to Y \subseteq Z)$, which contradicts the rigidity of Z. In the former case, the worldliness of W implies $\Box_{\mathbb{N}}\forall Y(WY \to \neg Yx)$ yielding the desired result. □

Now we can establish:

Lemma C.7 (Rigid Comprehension, Leibniz Biconditionals)**.** *Let* $\mathbf{R}$ *be any inflexible real number structure (e.g. as constructed above). For every second-order analytic statement,* $A(\mathbf{R})$*, with free first-order variables* $\vec{x} = x_1, \ldots, x_n$ *and free second-order variables* $\vec{X} = X_1, ..., X_k$*:*

$$\forall\vec{X}\forall\vec{x}((R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X}) \to A \to \Box_{\mathbb{N}}A)$$

where above we write $R\vec{x}$ *for* $Rx_1 \wedge \ldots \wedge Rx_n$*, and* $\vec{X} \subseteq R$ *to mean* $X_1 \subseteq R \wedge \ldots X_n \subseteq R$

Proof. We prove by induction on the structure of second-order analytic sentences, A that both A and its negation satisfy the theorem. Below .

1. $\forall\vec{X}\forall\vec{x}((R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X}) \to A \to \Box_{\mathbb{N}}A)$
2. $\forall\vec{X}\forall\vec{x}((R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X}) \to \neg A \to \Box_{\mathbb{N}}\neg A)$

Let $\vec{X}$ and $\vec{x}$ be arbitrary entities satisfying $(R\vec{x} \wedge \vec{X} \subseteq R \wedge \mathrm{Rig}\,\vec{X})$.

Atomic sentences have the form $x \leq y$, $x = y$, $x + y = z$, $Xy_1...y_n$, etc. 1 follows from the rigidity of the structure, in the former cases, or the rigidity of X in the last case. 2 follows from rigidity and the necessity of distinctness of $x, y, z, y_1...y_n$.

For conjunctions, suppose $A \wedge B$. We know from the inductive hypothesis that $\Box_{\mathbb{N}}A$ and $\Box_{\mathbb{N}}B$, so $\Box_{\mathbb{N}}(A \wedge B)$. This establish 1. in the case of 2, we have either $\neg A$ or $\neg B$, so by the inductive hypothesis one of these two claims is necessary, and thus so is $\neg(A \wedge B)$. For the negation case 1 is trivial from the IH, and 2 follows trivially from the IH and the equivalence of A and $\neg\neg A$.

For first-order generalizations. These have the form of a restricted quantification over the domain of $\leq$: $\forall x(Rx \to A)$. For 1, By the IH, for an arbitrary x in the domain, $\Box_{\mathbb{N}}A$, i.e. $\forall x(Rx \to \Box_{\mathbb{N}}A)$, so by the rigidity of R, we have $\Box_{\mathbb{N}}\forall x(Rx \to A)$. For 2, assume that universal is false: for some x, Rx and $\neg A$. We know that $\Box_{\mathbb{N}}\neg A$ by the inductive hypothesis, and we have $\Box_{\mathbb{N}}Rx$ by rigidity, so $\Box_{\mathbb{N}}\neg\forall x(Rx \to A)$, as required.

For second-order quantification we need Lemma C.6.

For 2, we will show the contrapositive. Suppose $\Diamond_{\mathbb{N}}\exists X(X \subseteq R \wedge \operatorname{Rig} X \wedge A)$. We wish to show $\exists X(X \subseteq R \wedge \operatorname{Rig} X \wedge A)$. Given the inductive hypothesis, it suffices to show $\exists X(X \subseteq R \wedge \operatorname{Rig} X \wedge \Diamond_{\mathbb{N}}A)$. Applying the Leibniz biconditionals to our assumption we get the existence of a world proposition, $\Box_{\mathbb{N}}\forall Y(WY \to (Y \subseteq R \wedge \operatorname{Rig} Y \wedge A[Y/X])$. By Lemma C.6, there is actually a rigid property, $X \subseteq R$ such that $\Box_{\mathbb{N}}\forall Y(WY \to X = Y)$, thus $\Diamond_{\mathbb{N}}(X \subseteq R \wedge \operatorname{Rig} x \wedge A)$.

□

Theorem C.3 (Rigid Comprehension, Leibniz Biconditionals)**.** *For any sentence of second-order analysis, A, with free variables $\mathbf{S}$, we can prove* $\Box\forall\mathbf{S}(\operatorname{Real}\mathbf{S} \to A) \vee \Box\forall\mathbf{S}(\operatorname{Real}\mathbf{S} \to \neg A)$*.*

Proof. The proof can be given as follows.

Suppose for contradiction that $\Diamond\exists\mathbf{S}(\operatorname{Real}\mathbf{S}\wedge A(\mathbf{S}))\wedge\Diamond\exists\mathbf{S}(\operatorname{Real}\mathbf{S}\wedge\neg A(\mathbf{S})$. Since there could have been a real number structure, the axiom of Potential Infinity is true, so $\mathbf{R}$ is an inflexible real number structure by Lemma C.4. Either $A(\mathbf{R})$ or $\neg A(\mathbf{R})$—without loss of generality, assume the former. Then we have that it's $\Box_{\mathbb{N}}$-necessary that $\mathbf{R}$ is a real number structure, by Lemma C.5, $\Box_{\mathbb{N}}$-necessary that $A(\mathbf{R})$ by Lemma C.7, and $\Box_{\mathbb{N}}$-necessary that $\forall\mathbf{S}(\operatorname{Real}\mathbf{S} \wedge \operatorname{Real}\mathbf{R} \to (A(\mathbf{R}) \leftrightarrow A(\mathbf{S}))$ by Lemma C.3. Thus $\Box_{\mathbb{N}}\forall\mathbf{S}(\operatorname{Real}\mathbf{S} \to A(\mathbf{S}))$. But $\Diamond\exists\mathbf{S}(\operatorname{Real}\mathbf{S}\wedge\neg A(\mathbf{S}))$ entails $\Diamond_{\mathbb{N}}\exists\mathbf{S}.\neg A(\mathbf{S})$, contradiction. In the case that $\neg A(\mathbf{R})$ the argument is similar.

□

References

Andrew Bacon. *A Philosophical Introduction to Higher-Order Logics*. Routledge, 2023.

Cian Dorr. To be f is to be g. *Philosophical Perspectives*, 30(1):39–134, 2016. doi: 10.1111/phpe.12079.

Zach Goodsell and Juhani Yli-Vakkuri. *Logical Foundations of Philosophy*. MS.

Harold T. Hodes. Where do the cardinal numbers come from? *Synthese*, 84 (3):347–407, 1990. doi: 10.5840/jphil198380supplementtoissue105.

A. N. Prior. Modality and quantification in s5. *Journal of Symbolic Logic*, 21(1):60–62, 1956. doi: 10.2307/2268488.

Arthur N. Prior. *Formal Logic*. Oxford University Press, 1955.